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Master thesis

A General Framework for Kinds of Forgetting in Knowledge Representation

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1 Introduction

In the past decade, the popularity and presence of artificial intelligence (AI) grew rapidly and thereby reached almost every part of our daily lives. From product and media recommendations, voice assistants, and smart homes over industrial optimizations, medical research, and traffic, to even criminal prosecution. And most probably, the importance of AI will grow even further in the near future, due to the ever-increasing amount of data that accumulates day by day and the huge potential it carries. Even though, AI allows us to optimize solutions and solve problems that were not possible to solve before, it also influences society as such and faces us with new challenges we have to meet, concerning the way we communicate, inform ourselves, consume, and last but not least how we evolve into a fairer society. Thus, whenever we are using AI, it is most important to think about its potential impacts. In the following, we want to picture two examples in which AI had a clearly negative impact on society.

In their report *Wrongfully Accused by an Algorithm*[1], Kashmir Hill from the New York Times covers the case of Robert Julian-Borchak Williams, who was wrongfully arrested due to an erroneous facial recognition. Joy Buolamwini from the Massachusetts Institute of Technology (MIT) Media Lab was able to show that the facial recognition model is biased towards certain demographic groups, such that its error rate is rather small as long as you are a white man, otherwise it rises up to 35 percent. Thus, the model structurally discriminates non-white and non-male people[2].

Another example for a negative impact of AI is YouTube's video recommendation model. The goal of this model is to recommend new and interesting videos for users, such that they spend more time on the platform. For a considerable time it was suspected that the model learned that it is most efficient to recommend more radical content over time [3,4], often concerning intended disinformation, hate speech and conspiracy beliefs. Nocun and Lamberty covered this problem among others in their book *Fake Facts* [NL20], and Ribeiro et al. recently investigated the radicalization effects of YouTube's recommendation model in [ROW+20], where they showed that there is significant evidence that the model directs you towards more extreme content, when starting on official news and media channels.

These two examples clearly show that it is not only sufficient to develop models that perform well in certain tests, but it is also necessary to gain a much deeper understanding of why models come to certain conclusions, and how undesired behaviour can be changed afterwards. This brings us to the concept of forgetting, which is also of particular interest since the General Data Protection Regulation

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[1] Wrongfully Accused by an Algorithm, by Kashmir Hill, The New York Times. (accessed January 12th 2021, 10:02 AM)

[2] Facial Recognition Is Accurate, if You're a White Guy, by Joy Buolamwini, MIT Media Lab, and Steve Lohr, The New York Times. (accessed January 13th 2021, 09:04 AM)

[3] Mit zwei Klicks in die Filterblase, by Philip Banse, Deutschlandfunk Kultur. (accessed January 13th 2021, 09:41 AM)

[4] 'Fiction is outperforming reality': how YouTube's algorithm distorts truth, by Paul Lewis, The Guardian. (accessed January 13th 2021, 09:45 AM)

(GDPR), which became applicable in 2018, gives every citizen of the European Economic Area the right to be forgotten (GDPR - Article 17). Even though, each and everyone has an intuitive idea of what forgetting means, it is necessary to understand the concept of forgetting in detail in order to apply it in the domain of AI. In [EK19], Ellwart and Kluge presented different psychological perspectives on forgetting and highlighted the importance of the collaboration between psychology and AI. The there presented perspectives on forgetting can be summarized as forgetting on the individual, the collective, and the organisational level. Individual forgetting only concerns the beliefs of a single individual, while collective and organisational forgetting argues about forgetting in a group of individuals. The latter can be further distinguished by saying that collective forgetting argues about the knowledge of the individuals in the group and the knowledge that can be considered as commonsense within this group, whereas forgetting in organisations concerns the knowledge about certain behaviour and routines with respect to the organisational objectives instead. Furthermore, Ellwart and Kluge distinguish between intentional and unintentional forgetting. While unintentional forgetting corresponds to the intuitive idea of forgetting as the unregulated loss of information, intentional forgetting describes the active process of removing undesired information, which generally goes back to personal motives.

In this thesis, we will focus on individual intentional forgetting, and examine it in the context of knowledge representation. The history of forgetting in knowledge representation goes back to the work of Boole [Boo54], which was already published in the year 1854. There they defined the syntactical forgetting of atoms in single propositions. Based on this, several other logic-specific forgetting approaches followed. In 2017, Delgrande [Del17] presented a general approach of forgetting with the goal to express the several existing logic-specific forgetting approaches, e.g. forgetting in first-order logic [LR94], answer set programming [Won09, ZF06], or propositional logic [Boo54], by means of a single definition. In contrast to the existing logic-specific approaches, Delgrande considers forgetting on a semantic level, and therefore is independent of the syntactic appearance of the knowledge, which in this case is represented by a set of formulas. Besides Delgrande, Kern-Isberner et al. [BKIS+19] also focus on elaborating a general framework for forgetting, but other than Delgrande they focus on the axiomatization of the different kinds of forgetting, which underlie certain cognitive considerations and are performed on epistemic states instead. With these two works being the most recent and promising approaches towards a general framework for forgetting, they form the foundations this thesis is based on.

The goal of this thesis is the attempt of elaborating forgetting postulates that capture the general properties of the different kinds of forgetting, and therefore are beyond those already stated in [BKIS+19]. These postulates are based on the properties Delgrande states for their general approach in [Del17]. We think that these properties are a good starting point for developing such postulates, since Delgrande's approach is already capable of expressing several of the hitherto logic-specific approaches. However, since these properties are specifically stated with respect to their definition of forgetting, we will generalize them, such that they are applicable to arbitrary belief change operators and epistemic states, instead of sets of formulas. Concretely, we will state two different sets of forgetting postulates. The first will state the properties of forgetting signature elements, whereas the second will state the properties of forgetting formulas. We think that this differentiation is necessary, since these two kinds of forgetting are conceptually different. On an intuitive level, forgetting signature elements corresponds to the idea of forgetting about the existence of certain objects and concepts of our worlds, while forgetting formulas corresponds to forgetting about specific facts about the latter. Thereby, we say that after forgetting a signature element, we do not want to be able to infer any propositions arguing about this element. On the other hand, when we forget a formula, we do not want to be able to infer this specific formula afterwards. Since we assume a purely propositional framework in this work, we argue that these two kinds of forgetting are exhaustive. Nonetheless, there might exist more kind of forgetting when working in a non-propositional framework. Furthermore, we will investigate the relations of these forgetting postulates to those already established in the domain of knowledge representation. This includes the generalized AGM postulates for epistemic states as presented in [KP17] and [DP97], and the postulates for iterated belief revision [DP97]. In addition, we will pursue the question, whether Delgrande's definition is already covered by the kinds of forgetting presented in [BKIS^{+}19]. Finally, we will discuss, whether the here elaborated postulates are really suitable for describing general properties of forgetting.

In the following, we state the structure of this thesis. In Section 2, we will define all the preliminaries that are needed in this work, including a brief introduction to propositional logic (Section 2.1), model theory and deductive reasoning (Section 2.2), as well as AGM theory and some epistemic terms (Section 2.3). In Section 2.4, we will present ordinal conditional functions (OCFs), which are a common choice for epistemic states in knowledge representation, their ability to handle uncertain knowledge, and the special relevance of minimal models. There we also state the relation between OCFs and faithfully assigned preorders, which are a common assumption for general epistemic states. After this, we will present and elaborate the different kinds of forgetting in Section 3, which are also covered later in Section 4. In Section 3.1, we present the general forgetting approach as presented by Delgrande in [Del17], alongside some model theoretical considerations and its most relevant properties, which form the basis for the here presented attempt of postulating general properties of forgetting. We also illustrate how Delgrande's forgetting approach can be used to express forgetting in propositional logic [Boo54]. Afterwards, we present three kinds of forgetting as presented by Kern-Isberner et al., namely the marginalization (Section 3.2.1), the contraction (Section 3.2.2), and the revision (Section 3.2.3), all with respect to OCFs. We decided to focus on these three kinds of forgetting for different reasons. The marginalization is the only kind of forgetting arguing about forgetting signature elements, and thus is most similar to Delgrande's approach. The contraction and revision can be considered as forgetting formulas, and form two of the three fundamental kinds of belief change according to AGM theory, which are subject of many researches in the domain of knowledge representation. Thus, we think that it is most important to cover both of them in this work as well. Besides the elaborations from Section 3, Section 4 forms the main part of this thesis. In Section 4.1, we compare the marginalization to Delgrande's approach and examine, whether both approaches result in equivalent beliefs. Furthermore, we generalize the properties of Delgrande's definition of forgetting to epistemic states, and therefore state the postulates for forgetting signature elements. We also examine, if the marginalization satisfies these postulates, and if there exist further operations that are capable of doing so. In Section 4.2, we present our attempt of postulating general properties of forgetting formulas and further examine, if they are satisfied by general contractions or those that only induce minimal changes to the prior beliefs. Furthermore, we investigate the relations between the AGM contraction postulates for epistemic states and the here stated forgetting postulates. Finally, we examine, if contractions are capable of expressing Delgrande's forgetting by means of literal forgetting. In Section 4.3, we consider revisions as a kind of forgetting and examine, if they are capable of satisfying the forgetting postulates. Furthermore, we also investigate the relations between the forgetting postulates and the AGM revision postulates for epistemic states, as well as the postulates for iterated revision as presented by Darwiche and Pearl [DP97]. In Section 4.4, we will discuss several controversial properties implied by the here presented forgetting postulates, and present suggestions how they should be adjusted in future works. Lastly, we will summarize our results alongside some open question that could be covered in future researches in Section 5. Note that all postulates frequently used in this thesis can also be found in the appendix Appendix A.1 for a much easier and faster access.

2 Preliminaries

In this section, we define all the preliminaries necessary for the different definitions and kinds of forgetting presented in Section 3, and moreover for the elaboration of a general framework for kinds of forgetting in Section 4. Since we generally assume a propositional framework in the later sections, we will first give a brief introduction to propositional logic in Section 2.1. There, we will define the general syntactic components of propositional logic, alongside the satisfaction relation and the basic propositional equivalences, which together form the semantics of propositional logic. Furthermore, we define the disjunctive and conjunctive normal form, which allow us to argue about propositions more easily, since they are guaranteed to match a certain pattern.

In Section 2.2, we present the basic definitions of model theory and deductive reasoning. This is of particular interest, since the forgetting approaches discussed in this work perform forgetting on the semantic level only. Thus, the syntactic structure of a formula is irrelevant for the result of forgetting. Furthermore, we define the concept of deductive reasoning, by means of the Tarskian consequence relation [Tar02] and the Cn operator [Mak88]. These will be necessary for the general forgetting approach presented by Delgrande in [Del17], but will also be used to argue about certain properties of the other kinds of forgetting discussed in this work.

Afterwards, we present some fundamental epistemic terms and give a brief introduction to AGM theory [AGM85, Mak85, Gär88, GR95] in Section 2.3. The presented overview of fundamental epistemic terms and their relations to each other makes it easier for the reader to thematically classify the different kinds of forgetting presented in the later sections. However, they will also be necessary to understand the there introduced AGM theory, which forms the basis for many researches in the domain of belief change and knowledge representation. We present the three basic kinds of belief change stated by AGM, namely the expansion, contraction and revision, and how they relate to each other, by means of postulates and identities. Both the postulates and the identities originally argue about belief changes in belief sets, where most forgetting approaches regarded in this work argue about epistemic states. Since the elaboration of their relations to the AGM postulates is a major part of this work, we present generalized versions of both the postulates and the identities, arguing about epistemic states, instead of belief sets. These generalized postulates were elaborated by Darwiche and Pearl in [DP97] and Konieczny and Pérez in [KP17], and allow us to examine the above-mentioned relations.

Lastly, we present ordinal conditional functions (OCFs) [Spo88] in Section 2.4, which form a common choice for epistemic states in the domain of knowledge representation, especially because of their capability of handling uncertain knowledge. In this work, OCFs are crucial for the kinds of forgetting presented by Kern-Isberner et al. [BKIS$^+$19], which form the major basis for our examinations, since they all argue about forgetting in OCFs. Beside the basic definitions and properties of OCFs, we further highlight some properties regarding minimal models, which are particularly important when arguing about belief changes in OCFs.

2.1 Propositional Logic

Propositional logic is one of the most fundamental logics and forms the basis for many others, such as first-order or modal logic. Even though one of the main approaches of forgetting discussed in this work can generally be applied to almost arbitrary logics, we will focus on propositional logic. Thus, we will only give an intuitive explanation on what logics in general are, since we think that a detailed elaboration of the latter is not necessary for this work. In the following, we will define the syntax and semantics of propositional logic and how they are related to each other. Further, we will define the basic equivalences that hold in propositional logic, which can also be used for syntactically transforming propositional formulas. Finally, we take a look at the conjunctive and disjunctive normal form, that will allow us to argue about formulas more easily by guaranteeing that they match a uniform structure. This is particularly useful when arguing about propositions with respect to their syntactic appearance.

First, we want to describe fundamental commonalities that most logics share. In general, a logic consists of three main components. The first component is the signature. It contains all the atomic elements the logic argues about. Mostly, these atomic elements have extra-logical meanings that describe how they can be understood. Intuitively speaking, a signature contains atomic elements corresponding to objects and concepts of our world, and further allows us to argue about them logically. The second component is a set of interpretations. Each interpretation maps the signature elements to certain (truth) values. These values depend on the choice of the specific logic. In classical logic, we generally have only two truth values – *true* and *false*, but there also exist logics with more than two values. So, interpretations can be understood as different perspectives on the objects and concepts of our world. The third component is the language. It contains more complex statements about the signature elements. These complex statements are formed by combining the atomic elements by means of certain junctors, which again depend on the specific logic. In addition to the signature, the interpretations, and the language, a relation is needed in order to state whether a sentence of the language is coherent with a given interpretation. The signature and the language form the logic's syntax, while the interpretations and the above-mentioned relation form its semantics.

In the following, we focus on propositional logic and start with the definition of its syntactic components. As already stated above for logics in general, the atomic components the formulas of a logic can be formed of are given by a signature. In propositional logic, the signature is a set of atomic propositions, which we will also refer to as atoms (Def. 2.1).

Definition 2.1. [BKI19] *A* signature Σ *is a finite set of atoms* $\rho \in \Sigma$*.*

By means of the junctors of propositional logic, we can form more complex propositions from the atoms given in the signature. The resulting set of propositions is called language. In Def. 2.2, we state the junctors of propositional logic and how they can be used to form the corresponding propositional language.

Definition 2.2. [BKI19] *Let* Σ *be a signature. The* language $\mathcal{L}_\Sigma$ *in propositional logic consists of the following propositions:*

- *Each* $\rho \in \Sigma$ *represents an* atomic proposition *in* $\mathcal{L}_\Sigma$*.*
- *For each proposition* $\varphi \in \mathcal{L}_\Sigma$*, the* negation $\neg\varphi$ *is also included in* $\mathcal{L}_\Sigma$*.*
- *For all propositions* $\varphi, \psi \in \mathcal{L}_\Sigma$*, the* conjunction $\varphi \wedge \psi$ *is also included in* $\mathcal{L}_\Sigma$*.*
- *For all propositions* $\varphi, \psi \in \mathcal{L}_\Sigma$*, the* disjunction $\varphi \vee \psi$ *is also included in* $\mathcal{L}_\Sigma$*.*

Atomic propositions and their negations are also often referred to as *literals*. Besides the negation $\neg$, conjunction $\wedge$ and disjunction $\vee$, there exist two more junctors in propositional logic, namely the material implication $\rightarrow$ and the co-implication $\leftrightarrow$. Both of these operators are not included in Def. 2.2, since they are only abbreviations for often used syntactical patterns. Therefore, they can also be expressed by means of $\neg$, $\wedge$ and $\vee$. Instead of $\neg\varphi \vee \psi$ we write $\varphi \rightarrow \psi$, and instead of $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ we write $\varphi \leftrightarrow \psi$. The interpretations of a propositional language then assign a truth value to each atom $\rho \in \Sigma$ (Def. 2.3).

Definition 2.3. [BKI19] *Let* Σ *be a signature, then* Ω_Σ *contains all corresponding* interpretations ω *that map each* $\rho \in \Sigma$ *to either* true *or* false*.*

In case of propositional logic, the assigned truth values are either *true* or *false*. Interpretations are often denoted by $\omega = \sigma\overline{\rho}$, where $\Sigma = \{\sigma, \rho\}$. Thereby, each overlined signature element is mapped to *false*, while the remaining are mapped to *true*. In order to assign a truth value to a non-atomic proposition $\varphi \in \mathcal{L}_\Sigma$, we further need to know about the behaviour of the junctors towards their interpretation. This behaviour is an integral part of propositional logic. The negation inverts the truth value of a proposition (Tab. 1). If a proposition is *true*, its negation is *false* and vice-versa. The conjunction of two propositions $\varphi \wedge \psi$ combines the truth values of φ and ψ such that it is *true*, if both values are *true*, and *false* otherwise. The disjunction of two proposition $\varphi \vee \psi$ combines the truth values of φ and ψ such that it is *true* if at least one of the values is *true*, and *false* otherwise. This is not to be confused with the *exclusive or* generally used in natural language, in which exactly one of the propositions must be *true*.

φ	$\neg\varphi$
false	*true*
true	*false*

Table 1: Semantic interpretation of the negation $\neg$.

φ	ψ	$\varphi \wedge \psi$
false	*false*	*false*
false	*true*	*false*
true	*false*	*false*
true	*true*	*true*

Table 2: Semantic interpretation of the conjunction $\wedge$.

φ	ψ	$\varphi \vee \psi$
false	*false*	*false*
false	*true*	*true*
true	*false*	*true*
true	*true*	*true*

Table 3: Semantic interpretation of the disjunction $\vee$.

By means of the interpretation of the atomic propositions and the given behaviour of the junctors, the truth value of a proposition can be determined by applying the above stated behaviour recursively. Given a certain interpretation, the truth assignment of a proposition is formally given by the satisfaction relation $\models$ (Def. 2.4).

Definition 2.4. [BKI19] *A* satisfaction relation $\models$ *relates interpretations* $\omega \in \Omega_\Sigma$ *to propositions* $\varphi \in \mathcal{L}_\Sigma$*, where* ω *satisfies* φ*, denoted by* $\omega \models \varphi$*, if and only if* φ *is* true *under the interpretation of the signature elements given by* ω*.*

Thereby, $\omega \models \varphi$ means that φ is *true* given the interpretation ω, while $\omega \not\models \varphi$ means that φ is *false* given ω. In Prop. 2.5, we state the basic semantic equivalences that hold in propositional logic.

Proposition 2.5. [BKI19] *Let* $\varphi, \psi, \xi \in \mathcal{L}_\Sigma$ *be propositions and* ω *an interpretation. The following equivalences hold in propositional logic.*

$$\begin{aligned} &\omega \models \varphi \Leftrightarrow \omega \models \varphi \vee \varphi \\ &\omega \models \varphi \Leftrightarrow \omega \models \varphi \wedge \varphi \end{aligned} \qquad \text{(Idempotence)}$$

$$\begin{aligned} &\omega \models \varphi \wedge \psi \Leftrightarrow \omega \models \psi \wedge \varphi \\ &\omega \models \varphi \vee \psi \Leftrightarrow \omega \models \psi \vee \varphi \end{aligned} \qquad \text{(Commutativity)}$$

$$\begin{aligned} &\omega \models (\varphi \wedge \psi) \wedge \xi \Leftrightarrow \omega \models \psi \wedge (\varphi \wedge \xi) \\ &\omega \models (\varphi \vee \psi) \vee \xi \Leftrightarrow \omega \models \psi \vee (\varphi \vee \xi) \end{aligned} \qquad \text{(Associativity)}$$

$$\begin{aligned} &\omega \models \varphi \wedge (\varphi \vee \psi) \Leftrightarrow \omega \models \varphi \\ &\omega \models \varphi \vee (\varphi \wedge \psi) \Leftrightarrow \omega \models \varphi \end{aligned} \qquad \text{(Absorption)}$$

$$\begin{aligned} &\omega \models \varphi \wedge (\psi \vee \xi) \Leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \xi) \\ &\omega \models \varphi \vee (\psi \wedge \xi) \Leftrightarrow (\varphi \vee \psi) \wedge (\varphi \vee \xi) \end{aligned} \qquad \text{(Distributivity)}$$

$$\omega \models \varphi \Leftrightarrow \omega \models \neg\neg\varphi \qquad \text{(Double negation)}$$

$$\begin{aligned} &\omega \models \neg(\varphi \vee \psi) \Leftrightarrow \omega \models \neg\varphi \wedge \neg\psi \\ &\omega \models \neg(\varphi \wedge \psi) \Leftrightarrow \omega \models \neg\varphi \vee \neg\psi \end{aligned} \qquad \text{(De Morgan's law)}$$

The idempotence states that the conjunction or disjunction of a proposition φ with itself is equivalent to φ. The commutativity and the associativity describe that conjunctions or disjunctions act indifferently to the order in which they are applied. The absorption property shows that we can simplify propositions of the form $\varphi \wedge (\varphi \vee \psi)$ or $\varphi \vee (\varphi \wedge \psi)$, since their truth values only depend on φ. If φ is *true* in $\varphi \wedge (\varphi \vee \psi)$, the truth value of ψ becomes irrelevant, since the proposition is *true* for both interpretations of ψ. If φ is *false* in $\varphi \wedge (\varphi \vee \psi)$, then the proposition cannot become *true* anymore because of the conjunction. Therefore, the truth value of ψ is again irrelevant. Analogously, this holds for $\varphi \vee (\varphi \wedge \psi)$, too. The distributivity describes that the conjunction of a formula with multiple disjunctively combined formulas is equivalent to a component-wise conjunction. The same holds for the disjunction analogously. The double negation states that negating the same proposition twice will be equivalent to the original proposition. Due to the binary truth value assignment, the negation always results in a proposition with opposite truth value. Applying the negation twice, we obtain the original truth value again. De Morgan's law describes the negation of a conjunction or disjunction of multiple propositions, which is similar to the idea of the distributivity. The negation will also be applied component-wise, but the disjunctions must be changed to conjunctions and vice-versa. This can easily be comprehended using the truth tables Tab. 1 to 3 above. We want to give a small example on the satisfaction relation in Ex. 2.1. Note that the there introduced signature Σ_{Tweety} will be repeatedly used for the following examples in this work.

Example 2.1. *In this example, we illustrate the satisfaction relation defined in Def. 2.4. For this, we first state the signature* $\Sigma_{Tweety} = \{f, b, p\}$ *with extra-logical meanings:*

$$\begin{aligned} &f \text{ – the observed animal can fly,} \\ &b \text{ – the observed animal is a bird,} \\ &p \text{ – the observed animal is a penguin.} \end{aligned}$$

Furthermore, let $\mathcal{L}_{\Sigma_{Tweety}}$ *be the propositional language over* Σ_{Tweety} *and*

$$\omega_1 = p\overline{f}\overline{b}, \quad \omega_2 = p\overline{b}f, \quad \omega_3 = pbf, \quad \omega_4 = \overline{p}\overline{b}f.$$

are corresponding interpretations in $\Omega_{\Sigma_{Tweety}}$. *For* ω_1*-*ω_4 *the following relations hold:*

$$\begin{aligned} \omega_1 &\models p \\ \omega_2 &\not\models \neg p \vee \neg f \\ \omega_3 &\models (p \to b) \wedge (b \to f) \\ \omega_4 &\models \{b \to f, \neg p \vee \neg f\} \end{aligned}$$

Two further important properties of propositional logic are that syntactically transforming a proposition preserves its truth value, and that if two formulas are satisfied by the same interpretations they can also obtained from each other by means of syntactic transformations. These properties are better known as *soundness* and *completeness*. Therefore, the equivalences stated above in Prop. 2.5 can also be used for syntactic transformations. If a proposition ψ can be obtained from another proposition φ by means of syntactically transforming φ, we also say that φ syntactically infers ψ, which is denoted by $\varphi \vdash \psi$.

When arguing about propositional logic, a problem that often occurs is that the syntactic structure of a proposition can be almost arbitrary. This makes it difficult to argue about propositions when their syntactic structure is of importance, e.g. for developing algorithms or complexity analysis. Therefore, the concept of normal forms is particularly important. A normal form specifies a uniform structure each proposition can take on by means of syntactical inference. In the following, we will introduce two of the most well-known normal forms in propositional logic – the conjunctive and the disjunctive normal form.

A proposition in conjunctive normal form (CNF) consists of disjunctive clauses in which all propositions are literals, i.e. a positive or negative atomic proposition. All those disjunctive clauses are combined conjunctively. Formally, we define the CNF as given in Def. 2.6.

Definition 2.6. [BKI19] *A proposition $\varphi \in \mathcal{L}_\Sigma$ is in* conjunctive normal form (CNF)*, if and only if it is of the form*

$$(\lambda_{1,1} \vee \cdots \vee \lambda_{1,m_1}) \wedge \cdots \wedge (\lambda_{n,1} \vee \cdots \vee \lambda_{n,m_n}),$$

where $(\lambda_{i,j})_{i,j \in \mathbb{N}_0}$ are literals.

Analogously to the CNF, we can define the disjunctive normal form (DNF) in Def. 2.7.

Definition 2.7. [BKI19] *A proposition $\varphi \in \mathcal{L}_\Sigma$ is in* disjunctive normal form (DNF)*, if and only if it is of the form*

$$(\lambda_{1,1} \wedge \cdots \wedge \lambda_{1,m_1}) \vee \cdots \vee (\lambda_{n,1} \wedge \cdots \wedge \lambda_{n,m_n}),$$

where $(\lambda_{i,j})_{i,j \in \mathbb{N}_0}$ are literals.

In case of the DNF, the literals in the clauses are combine conjunctively, while the clauses are combined disjunctively. Further, we know that each proposition can be transformed into an equivalent proposition (Prop. 2.8) in CNF or DNF using the syntactic rules in Prop. 2.5. Thus, propositions can always be assumed to be in CNF or DNF if necessary.

Proposition 2.8. [BKI19] *For each proposition $\varphi \in \mathcal{L}_\Sigma$ there exists an equivalent proposition $\psi \in \mathcal{L}_\Sigma$ that is in conjunctive normal form or disjunctive normal form.*

In summary, we gave an intuitive explanation of logics in general, and further introduced the basic definitions of propositional logic. This included among others the syntactic components, namely the signature and the language, as well as the semantic components, namely the interpretations and the satisfaction relation. Furthermore, we stated the basic equivalences that hold in propositional logic and presented the conjunctive and disjunctive normal form, which can be used to guarantee a uniform syntactic structure of propositions, and therefore make arguing about them easier.

2.2 Model Theory and Deductive Reasoning

In this section, we will introduce the basic definitions and properties of model theory as needed in this work, as well as the concept of deductive reasoning, which describes the logical inference of knowledge from a given set of formulas without further assumptions. In contrast to syntactic inference as presented in Section 2.1, deductive reasoning is purely semantic. Thus, the syntactic appearance of a formula is irrelevant for the reasoning. Instead, it argues about the interpretations that satisfy the formula, i.e. its models. Since deductive reasoning forms the most fundamental kind of reasoning in the domain of knowledge representation, it will be essential for the examinations in this work, too. Furthermore, we will show that deductive reasoning is closely related to the concept of theory. Lastly, we state several equivalences regarding the models of a formula, its deductive closure and the corresponding theory. These equivalences emphasize the relations between those three concepts, and further allow us to argue about them more easily in the later sections.

First, we like to introduce the definition of models in Def. 2.9.

Definition 2.9. [BKI19] *The* models $[\![\varphi]\!]_\Sigma$ *of a formula* $\varphi \in \mathcal{L}_\Sigma$ *with respect to the signature* Σ *contains all interpretations* $\omega \in \Omega_\Sigma$ *satisfying* φ*.*

$$[\![\varphi]\!]_\Sigma = \{\omega \in \Omega_\Sigma \mid \omega \models \varphi\}$$

The models of a formula $\varphi \in \mathcal{L}_\Sigma$ consists of all interpretations in Ω_Σ that satisfy φ. Generally, the corresponding signature Σ is written in the subscript of the model brackets, but it can also be omitted if the signature is clearly given by the context. This will be of particular importance in the later sections, where we argue about models in different signatures. There, the subscript clarifies the corresponding signatures. From the semantics of $\wedge, \vee$ and $\neg$ as presented in Section 2.1, we can derive the following behaviour for the models of a conjunction, disjunction and negation (Lem. 2.10).

Lemma 2.10. *Let* $\varphi, \psi \in \mathcal{L}_\Sigma$ *be formulas and* Ω_Σ *the set of corresponding interpretations, then the following relations hold:*

$$\begin{aligned} [\![\varphi \vee \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\neg\varphi]\!] &= \Omega_\Sigma \setminus [\![\varphi]\!] \end{aligned}$$

The models of a disjunction $\varphi \vee \psi$ consists of both the models of φ and the models of ψ, since $\varphi \vee \psi$ is already satisfied by an interpretations if either φ or ψ is satisfied. This holds similarly for a conjunction $\varphi \wedge \psi$, which is satisfied by an interpretation, if and only if it satisfies both φ and ψ. Thus, the models of a conjunction equals the intersection of the models of φ and ψ. The models of a negation $\neg\varphi$ are the complement of the models of φ, since each interpretation must either satisfy φ or $\neg\varphi$.

If we consider sets of formulas Γ and Γ' instead of single formulas, we can conclude that a model of $\Gamma \cup \Gamma'$ must be a model of both Γ and Γ' as well (Lem. 2.11).

Lemma 2.11. [BKI19] *Let $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ sets of formulas and $\omega \in \Omega_\Sigma$ an interpretation, then the following holds:*

$$\omega \models \Gamma \cup \Gamma' \Leftrightarrow \omega \models \Gamma \text{ and } \omega \models \Gamma'$$

By means of the model theoretical basics stated above, we can further define the concept of deductive reasoning, which is the most basic form of reasoning, since it only infers knowledge that directly concludes from a given set of formulas without making further assumptions. Transferring this notion to a logical level, we say that certain knowledge can be inferred by a given set of formulas Γ, if it is true, whenever Γ is true. Formally, this relation is defined by the Tarskian consequence relation (Def. 2.12).

Definition 2.12. [Tar02] *A* Tarskian consequence relation $\models$ *relates two sets of formulas $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ to each other, such that*

$$\Gamma \models \Gamma' \Leftrightarrow [\![\Gamma]\!]_\Sigma \subseteq [\![\Gamma']\!]_\Sigma.$$

Thereby, a set of formulas Γ' is said to be a logical consequence of Γ, denoted by $\Gamma \models \Gamma'$, if and only if all interpretations that satisfy Γ also satisfy Γ'. We also say that Γ' can be deductively inferred by Γ. This relation can also be applied to single formulas analogously. In case that Γ infers Γ' and vice-versa, we say that both sets of formulas are equivalent (Def. 2.13).

Definition 2.13. [BKI19] *Let $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ be two sets of formulas, then Γ and Γ' are* equivalent, *denoted by $\Gamma \equiv \Gamma'$, if and only if the following holds:*

$$\Gamma \equiv \Gamma' \Leftrightarrow (\Gamma \models \Gamma' \text{ and } \Gamma' \models \Gamma) \Leftrightarrow [\![\Gamma]\!]_\Sigma = [\![\Gamma']\!]_\Sigma$$

The definition of the equivalence of two (sets of) formulas (Def. 2.13) also provides an interesting relation between a formula φ and its models, which is the equivalence of φ and the disjunction of all its models $[\![\varphi]\!]$ (Prop. 2.14)

Proposition 2.14. *Let $\varphi \in \mathcal{L}_\Sigma$ be a formula, then the following holds:*

$$\varphi \equiv \bigvee_{\omega \in [\![\varphi]\!]} \omega$$

Proof of Prop. 2.14. From Def. 2.13, we know that the stated equivalence holds, if and only if the corresponding models are equal. We prove the equality of the models in the following:

$$\begin{aligned} [\![\varphi]\!] &= \bigcup_{\omega \in [\![\varphi]\!]} \{\omega\} \\ &= \bigcup_{\omega \in [\![\varphi]\!]} [\![\omega]\!] && ([\![\omega]\!] = \{\omega\}) \\ &= [\![\bigvee_{\omega \in [\![\varphi]\!]} \omega]\!] && (\text{Lem. 2.10}) \end{aligned}$$

□

At this point, we make use of the fact that a model $\omega \in [\![\varphi]\!]$ can also be viewed as a conjunction of literals, e.g. the interpretation $p\bar{b}f$ can also be viewed as $p \wedge \neg b \wedge f$. Thus, we make use of this *trick*, whenever we treat interpretations like formulas in the following.

In contrast to the semantic inference based on the Tarskian consequence relation, we already described syntactic inference in Section 2.1, which is based on the syntactic transformation of formulas. These two kinds of inferences are closely related through the properties of soundness and completeness [BKI19]. Soundness states that each conclusion that can be inferred syntactically can also be inferred semantically:

$$\text{If } \varphi \vdash \psi, \text{ then } \varphi \models \psi.$$

This means that the models of φ are models of ψ as well, for every ψ that can be obtained by syntactically transforming φ. In other words, it is not possible to infer some ψ syntactically that is not satisfied by the models of φ. Conversely, the correctness states that each conclusion that can be inferred semantically can also be inferred syntactically:

$$\text{If } \varphi \models \psi, \text{ then } \varphi \vdash \psi.$$

This means that all formulas ψ that are a consequence of φ can also be obtained through syntactic transformation. If both soundness and completeness hold, we know that these inferences are equivalent, yielding the exact same conclusions:

$$\varphi \vdash \psi \Leftrightarrow \varphi \models \psi.$$

Moreover, we want to state the relation between formulas φ, ψ and their negations in case that $\varphi \models \psi$ holds. Since we know from Lem. 2.10 that the models of a negation $\neg\varphi$ are complementary to the models of φ, we can conclude that if $\varphi \models \psi$ holds, then $\neg\psi \models \neg\varphi$ must hold as well and vice-versa (Lem. 2.15).

Lemma 2.15. *Let $\varphi, \psi \in \mathcal{L}_\Sigma$ be formulas, then the following holds:*

$$\varphi \models \psi \Leftrightarrow \neg\psi \models \neg\varphi$$

In many cases, it is necessary to know all formulas that can be inferred by a given set of formulas. Thus, we further define the consequence operator Cn in Def. 2.16 that maps a set of formulas to all its consequences by means of a Tarskian consequence relation.

Definition 2.16. [Mak88] *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas. The* consequence operator *Cn maps Γ to all formulas $\varphi \in \mathcal{L}_\Sigma$ with $\Gamma \models \varphi$.*

$$Cn(\Gamma) = \{\varphi \in \mathcal{L}_\Sigma \mid \Gamma \models \varphi\}$$

Since $Cn(\Gamma)$ contains *all* formulas that can be directly concluded from Γ, we say that $Cn(\Gamma)$ is deductively closed. In the following, we also write $Cn(\varphi_1, \ldots, \varphi_n)$ instead of $Cn(\{\varphi_1, \ldots, \varphi_n\})$ for formulas $\varphi_i \in \mathcal{L}$ and $i \in \mathbb{N}$. Due to the Tarskian consequence relation, we know that the models of $Cn(\Gamma)$ must be same as those of Γ, and thus a set of formulas is always equivalent to its deductive closure (Lem. 2.17).

Lemma 2.17. *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ a set of formulas, then the following holds:*

$$Cn(\Gamma) \equiv \Gamma$$

In the following, we state the main properties of the Tarskian consequence relation and the consequence operator in Th. 2.18 and 2.19. These allow us to argue about deductive reasoning on the formula level instead of the model level, which can often be understood more intuitively.

Theorem 2.18. [Mak88] *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\models$ a Tarskian consequence relation, then the following holds:*

$$\begin{aligned} &\textit{If } \varphi \in \Gamma, \textit{ then } \Gamma \models \varphi && \text{(Reflexivity)} \\ &\textit{If } \Gamma \models \varphi \textit{ and } \Gamma \cup \{\varphi\} \models \psi, \textit{ then } \Gamma \models \psi && \text{(Cut)} \\ &\textit{If } \Gamma \models \psi, \textit{ then } \Gamma \cup \{\varphi\} \models \psi && \text{(Monotony)} \end{aligned}$$

Theorem 2.19. [Mak88] *Let $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ be sets of formulas, and Cn a consequence operator, then the following holds:*

$$\begin{aligned} &\Gamma \subseteq Cn(\Gamma) && \text{(Reflexivity)} \\ &\textit{If } \Gamma \subseteq \Gamma' \subseteq Cn(\Gamma), \textit{ then } Cn(\Gamma') \subseteq Cn(\Gamma) && \text{(Cut)} \\ &\textit{If } \Gamma \subseteq \Gamma', \textit{ then } Cn(\Gamma) \subseteq Cn(\Gamma') && \text{(Monotony)} \end{aligned}$$

In Th. 2.19 it can easily be seen that the properties cut and monotony induce the cumulativity of logical consequences [Mak88], since whenever the condition of cut is fulfilled the condition of monotony is fulfilled as well:

$$\begin{array}{c} \text{If } \Gamma \models \varphi, \text{ then } \Gamma \models \psi \text{ iff } \Gamma \cup \{\psi\} \models \varphi, \\ \text{resp. if } \Gamma \subseteq \Gamma' \subseteq Cn(\Gamma), \text{ then } Cn(\Gamma') = Cn(\Gamma). \end{array} \qquad \text{(Cumulativity)}$$

Further, we will explain the properties stated in Th. 2.18 and 2.19 with respect to the consequence operator Cn only. However, the given explanations can easily be transferred to the Tarskian consequence operator.

The reflexivity ensures that a set of formulas $\Gamma \subseteq \mathcal{L}$ is always included in its own logical consequences $Cn(\Gamma)$. Since $Cn(\Gamma)$ consists of all formulas that are satisfied by the models $[\![\Gamma]\!]$, $Cn(\Gamma)$ in particular contains Γ itself, because $[\![\Gamma]\!] \models \Gamma$ holds by definition.

The cut property states that when adding a formula φ to the set of formulas Γ that could already be inferred, it does not affect the resulting deductive closure $Cn(\Gamma)$. Concretely, if we can infer $\psi \in \mathcal{L}$ from $\Gamma \cup \{\varphi\}$ deductively, but also φ from Γ, then adding φ to Γ does not give us any new information. Therefore, we can also infer ψ from Γ without adding φ.

The monotony states that after adding new information, we can still infer the same knowledge as before. At first glance, it may seem that this property does not hold when adding information that is inconsistent to the present knowledge. This is the case when we add a formula φ to a set of formulas Γ with $\Gamma \models \neg\varphi$.

By definition of the Tarskian consequence relation (Def. 2.12), we know that $\neg\varphi$ concludes deductively from Γ, if and only if its models are included in those of $\neg\varphi$:

$$\Gamma \models \neg\varphi \Leftrightarrow [\![\Gamma]\!] \subseteq [\![\neg\varphi]\!].$$

Since φ and $\neg\varphi$ are mutually exclusive, we further know

$$[\![\varphi]\!] \cap [\![\neg\varphi]\!] = \emptyset,$$

due to Lem. 2.10. When we then add φ to Γ, there will be no models that satisfy the resulting set of formulas, since there do not exist interpretations satisfying $\varphi \wedge \neg\varphi$. The models of $\Gamma \cup \{\varphi\}$ can be described by intersecting the model sets of Γ and φ, since the a set of formulas is satisfied, if and only if the conjunction of all formulas in this set is satisfied. Since $[\![\Gamma]\!] \subseteq [\![\neg\varphi]\!]$ and $[\![\varphi]\!] \cap [\![\neg\varphi]\!] = \emptyset$, we also know

$$[\![\Gamma \cup \{\varphi\}]\!] = [\![\Gamma]\!] \cap [\![\varphi]\!] = \emptyset.$$

At this point, we want state that the whole language $\mathcal{L}$ can be inferred from an contradiction $\perp$, since $[\![\perp]\!] = \emptyset$ (Lem. 2.20).

Lemma 2.20. *For each propositional language $\mathcal{L}_\Sigma$, the following holds:*

$$Cn(\perp) = \mathcal{L}_\Sigma$$

This directly concludes from the definition of the Tarskian consequence relation (Def. 2.12), in which it is stated that a formula φ can be deductively inferred from a set of formulas Γ, if and only if $[\![\Gamma]\!] \subseteq [\![\varphi]\!]$. Applying this definition to $\perp$, we obtain

$$\begin{aligned} Cn(\perp) &= \{\varphi \in \mathcal{L} \mid \perp \models \varphi\} && \text{(Def. 2.16)} \\ &= \{\varphi \in \mathcal{L} \mid [\![\perp]\!] \subseteq [\![\varphi]\!]\} && \text{(Def. 2.12)} \\ &= \{\varphi \in \mathcal{L} \mid \emptyset \subseteq [\![\varphi]\!]\} \\ &= \mathcal{L}. \end{aligned}$$

Therefore, the monotony holds even if we add contradictory knowledge to Γ. Finally, we state some equivalent relations in Prop. 2.21 summarizing the relations between model sets, the Tarskian consequence relation and the Cn operator.

Proposition 2.21. *Let $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ be sets of formulas, the the following relations are equivalent:*

1. $[\![\Gamma]\!] \subseteq [\![\Gamma']\!]$
2. $\Gamma \models \Gamma'$
3. $Cn(\Gamma) \models Cn(\Gamma')$
4. $Cn(\Gamma') \subseteq Cn(\Gamma)$

Next, we want to show the strong relation between the concept of theory and deductive reasoning. For this, we first give the definition of a theory in Def. 2.22.

Definition 2.22. *Let $\Theta \subseteq \Omega_\Sigma$ be a set of interpretations. The* theory *of Θ is defined as*

$$Th(\Theta) = \{\varphi \in \mathcal{L}_\Sigma \mid \Theta \models \varphi\}.$$

Intuitively, a theory describes all knowledge that is coherent with a given set of interpretations Θ. Formally, this is described by the set of all formulas that are satisfied by Θ. Just as the Cn operator, Th results in a deductively closed set of formulas. Moreover, the theory of a set of interpretations can also be described by means of Cn (Lem. 2.23). This can be traced back to the equivalence of a formula and the disjunction of its models stated in Prop. 2.14.

Lemma 2.23. *Let $\Theta \subseteq \Omega_\Sigma$ be a set of interpretations, then the following holds:*

$$Th(\Theta) = Cn(\bigvee_{\omega \in \Theta} \omega)$$

This allows us to use the deductive closure of a formula and the theory of its models interchangeably. Furthermore, we know that the models of a theory must be equal to the set of interpretations Th was applied to (Lem. 2.24).

Lemma 2.24. *Let $\Theta \subseteq \Omega_\Sigma$, then the following holds:*

$$[\![Th(\Theta)]\!] = \Theta$$

This also corresponds to the fact that the models of a deductive closure $Cn(\Gamma)$ are equal to the models of Γ. Another important property we will need in this work is that the intersection of two theories $Th(\Theta)$ and $Th(\Theta')$ can be described by the theory of $\Theta \cup \Theta'$ (Lem. 2.25).

Lemma 2.25. *Let $\Theta, \Theta' \subseteq \Omega_\Sigma$ be sets of interpretations, then the following relation holds:*

$$Th(\Theta) \cap Th(\Theta') \equiv Th(\Theta \cup \Theta')$$

Intuitively, Lem. 2.25 states that the formulas both theories $Th(\Theta)$ and $Th(\Theta')$ agree on are the formulas that can be inferred from both interpretation sets Θ, Θ'. To summarize the relations between a set of formulas, its deductive closure and the theory of its models, we state several equivalences in Prop. 2.26 below.

Proposition 2.26. *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas, then the following sets are equivalent:*

$$\Gamma \equiv Cn(\Gamma) \equiv Cn(\bigvee_{\omega \in [\![\Gamma]\!]} \omega) \equiv Th([\![\Gamma]\!])$$

In conclusion, we gave a short introduction to model theory by presenting the basic definitions and properties as needed in this work. Thereby, we answered the questions what models are, how they can be determined, and how they relate to their corresponding formulas. Moreover, we showed how deductive reasoning can formally be described by means of models. Thereby, we introduced the Tarskian consequence relation that determines, whether a formulas be can deductively inferred from another, as well as the consequence operator Cn that determines the deductively closed set of consequences of a formula. For both the relation and the operator we stated the most relevant properties including certain equivalences that emphasize the relations between models, formulas and their consequences. Lastly, we showed that the concept of deductive reasoning is strongly related to the concept of theory, and stated some of the basic properties and equivalences.

2.3 Fundamentals of Epistemology and AGM Theory

In this section, we will introduce the fundamental epistemic terms in knowledge representation and their relations to each other. These include among others the concepts of knowledge bases, belief sets and epistemic states. Thereby, we will briefly highlight the special significance of *faithful assignments* for the latter. In addition, we will state the three fundamental belief change operations in knowledge representation, namely the expansion, contraction and revision. These operations are best known in the context of AGM theory [AGM85, GR95], in which their general properties and connections to each other are given in the form of postulates and identities. We will present the latter as needed in this work, and furthermore their generalized forms for epistemic states. Note that all postulates presented in this section will be frequently used in the later sections. Therefore, they can also be found in Appendix A.1 for a faster and easier access.

Fundamental epistemic terms. We want to start with an overview of the most basic epistemic terms, and their relations to each other. This overview is also illustrated in Figure 1. The most basic of the epistemic terms is that of a *knowledge base*, which describes a set of secure or uncertain information that forms the basis of our knowledge. These information can be logical formulas or probabilities, for example. Given a certain knowledge base, we can obtain its corresponding *belief set* by means of deductive reasoning. Thereby, the belief set states the deductively closed set of conclusions that can be inferred from the knowledge base without further assumptions. In the context of propositional logic, we already introduced this concept by means of the Tarskian consequence relation (Def. 2.12) and the Cn operator (Def. 2.16) in Section 2.2. However, the knowledge base can also be used to obtain an *epistemic state*, that describes our knowledge in a more abstract manner. Commonly used examples for epistemic states are Bayesian networks [CH92], decision trees [JWHT13], and ordinal conditional functions [Spo88]. The latter will be introduced in Section 2.4. Epistemic states represent our knowledge without explicitly stating our beliefs, but by estimating probability functions, arranging certain rules, or ordering interpretations, depending on the kind of epistemic state. Generally, an epistemic state can be obtained from a knowledge base by means of inductive inference, which forms one of the three commonly considered kinds of reasoning, beside deductive and abductive reasoning [Pei65]. What distinguishes inductive from deductive inference, is that it infers general knowledge from more specific information. This always requires further assumptions that are not given by the knowledge base,

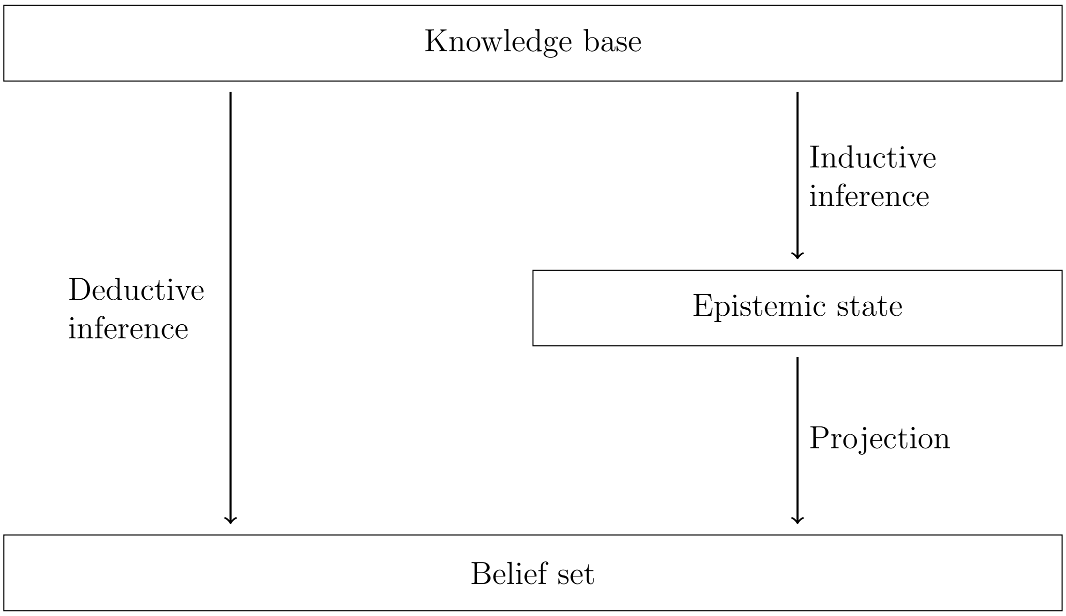


Figure 1: Relations of the epistemic terms *knowledge base*, *epistemic state* and *belief set*. This figure is adapted from the slides of the lecture *Commonsense Reasoning* read by Kern-Isberner at TU Dortmund University in summer 2019.

which in this case is used to evidence the more general knowledge. Note that this is conceptually different to inferring information from the knowledge base deductively. Information that are inferred inductively do not necessarily have to be correct, but can only be considered probable, since the knowledge base is just an evidence rather than a proof. An epistemic state can further be used to determine whether certain information are believed or not. Thus, we can obtain the belief set of an epistemic state by means of a *projection* function. The way the projection maps an epistemic state to its beliefs again depends on the kind of epistemic state chosen to represent our knowledge. It is even possible that there exist different projections for the same epistemic state. Note that the terms stated above are not exhaustive, but only state the very fundamental terms as needed in this work.

As described above, the concept of epistemic states is very general and rather abstract, and can be realized in many different ways. However, in the domain of knowledge representation, epistemic states Ψ are often assumed to be equipped with a faithfully assigned total preorder $\preceq_\Psi$ on the interpretations Ω, which allows an ordering of our knowledge, for example with respect to its probability or plausibility. The term *faithful* thereby states that the assigned total preorder must follow certain conditions. We state these conditions in Def. 2.27 below.

Definition 2.27. [DP97] *Let Ψ and Φ be epistemic states and Ω a set of interpretations that belong to them. A* faithful assignment *maps each epistemic state Ψ to a total pre-order $\preceq_\Psi \subseteq \Omega \times \Omega$ satisfying the following conditions for all $\omega, \omega' \in \Omega$:*

1. *$\omega, \omega' \models Bel(\Psi)$, only if $\omega =_\Psi \omega'$,*
2. *$\omega \models Bel(\Psi)$ and $\omega' \not\models Bel(\Psi)$, only if $\omega \preceq_\Psi \omega'$ and not $\omega' \preceq_\Psi \omega$,*

3. $\Psi = \Phi$, *only if* $\preceq_\Psi = \preceq_\Phi$,

where $\omega =_\Psi \omega'$ *is defined as* $\omega \preceq_\Psi \omega'$ *and* $\omega' \preceq_\Psi \omega$*; and* $\omega <_\Psi \omega'$ *is defined as* $\omega \preceq_\Psi \omega'$ *and* $\omega' \not\preceq_\Psi \omega$.

The three conditions a faithfully assigned total preorder $\preceq_\Psi$ must fulfil, can intuitively be understood as follows. Two interpretations satisfy the beliefs of an epistemic state, only if they are equal according to $\preceq_\Psi$. This means, that all models of $Bel(\Psi)$ are equal according to the total preorder, and if an interpretation is for example less probable or plausible than a model of $Bel(\Psi)$, then it cannot be a model of $Bel(\Psi)$. Lastly, two epistemic states can only be equal, if their assigned total preorders are equal as well. However, this does not exclude that two different epistemic states are assigned to the same total preorder.

Introduction to AGM theory. Next, we introduce the three fundamental belief change operations, namely the expansion, contraction and revision as elaborated in AGM theory [AGM85, GR95]. All of these operations argue about belief sets and formulas, and thereby follow the *minimum change paradigm*, which states that our beliefs should only be changed in a minimum way when performing certain belief changes. Concretely, this means that when removing or adding new beliefs to our belief set, only those formulas should be removed or added that are crucial for successfully performing the corresponding belief change. This consideration is also stated in the properties postulated by AGM. First, we present the expansion, denoted as $+$, which forms the simplest of the three belief change operations. The general properties of an expansion are given by the following postulates **(AGM+1)**-**(AGM+6)** [Mak85], where K, K' are belief sets and $\varphi \in \mathcal{L}$ is a formula:

(AGM+1) $K + \varphi$ is a belief set

(AGM+2) $\varphi \in K + \varphi$

(AGM+3) $K \subseteq K + \varphi$

(AGM+4) If $\varphi \in K$, then $K + \varphi = K$

(AGM+5) If $K \subseteq K'$, then $K + \varphi \subseteq K' + \varphi$

(AGM+6) $K + A$ is the smallest belief set, such that **(AGM+1)**-**(AGM+5)** hold

These postulates state that each expansion $K + \varphi$ should again result in a belief set **(AGM+1)**, in which the just added formula φ is included **(AGM+2)**. Thus, **(AGM+2)** is also called the success postulate. It is of particular importance that due to the expansion none of prior beliefs are rejected **(AGM+3)**. This means, that even if φ contradicts the prior beliefs K, none of them will be rejected in order to prevent a posterior contradictory belief set. **(AGM+5)** states the monotony that retains the subset relation of prior belief sets K, K' after an expansions with φ. This is strongly related to the underlying minimum change paradigm, which is more explicitly stated by **(AGM+4)** and **(AGM+6)**. Given the AGM expansion postulates **(AGM+1)-(AGM+6)**, we can state that an operator $+$ satisfies them, if and only if it equals the deductive closure of the prior beliefs K unified with φ (Th. 2.28).

Theorem 2.28. [GR95] *A belief change operator $+$ satisfies **(AGM+1)-(AGM+6)**, if and only if*

$$K + \varphi = Cn(K \cup \{\varphi\})$$

While the expansion as originally stated by AGM is sufficient for the elaborations in this work, we cannot make use of the originally stated contraction and revision, because they only argue about belief sets. Since the examinations towards a general framework for kinds of forgetting will mostly argue about forgetting in epistemic states, we need a generalized form of the AGM contraction and revision postulates that argue about epistemic states as well. This will allow us to examine how the properties of contractions and revisions relate to the forgetting postulates, which will be presented in the later sections. The postulates **(AGMes-1)-(AGMes-7)** as presented by Konieczny and Pérez in [KP17] state the AGM contraction postulates generalized to epistemic states, where Ψ is an epistemic state, $\varphi, \psi \in \mathcal{L}$ are formulas, and $-$ a belief change operator over epistemic states:

(AGMes-1) $Bel(\Psi) \models Bel(\Psi - \varphi)$

(AGMes-2) If $Bel(\Psi) \not\models \varphi$, then $Bel(\Psi - \varphi) \models Bel(\Psi)$

(AGMes-3) If $Bel(\Psi - \varphi) \models \varphi$, then $\varphi \equiv \top$

(AGMes-4) $Bel(\Psi - \varphi) \cup \{\varphi\} \models Bel(\Psi)$

(AGMes-5) If $\varphi \equiv \psi$, then $Bel(\Psi - \varphi) \equiv Bel(\Psi - \psi)$

(AGMes-6) $Bel(\Psi - \varphi \wedge \psi) \models Bel(\Psi - \varphi) \vee Bel(\Psi - \psi)$

(AGMes-7) If $Bel(\Psi - \varphi \wedge \psi) \not\models \varphi$, then $Bel(\Psi - \varphi) \models Bel(\Psi - \varphi \wedge \psi)$

A contraction $\Psi - \varphi$ describes a belief change that removes certain formulas from our prior beliefs $Bel(\Psi)$ without adding new information, such that φ is not believed by the posterior beliefs **(AGMes-1)**. However, this does not hold if φ is a tautology. In this case, φ is still included in the posterior beliefs, due to the nature of tautologies **(AGMes-3)**. Furthermore, if φ was not included in the prior beliefs in the first place, then the contraction will not reduce the prior beliefs **(AGMes-2)**. This again corresponds to the underlying minimum change paradigm. Moreover, the contraction is independent of the syntactic structure of the contracted formulas, which concludes that contracting two equivalent formulas $\varphi \equiv \psi$ will also result in equivalent beliefs **(AGMes-5)**. The beliefs that were rejected due to a contraction can be recovered by adding φ to the posterior beliefs $Bel(\Psi - \varphi)$ **(AGMes-4)**. Thus, **(AGMes-4)** is also called the recovery postulate. **(AGMes-6)** states that contracting a conjunction $\varphi \wedge \psi$ only removes those formulas from the prior beliefs that are necessary to guarantee that $\varphi \wedge \psi$ cannot be inferred by the posterior beliefs. Concretely, this means that it is generally sufficient to contract the beliefs that are either related to φ or ψ. **(AGMes-7)** states that if φ cannot be inferred by the $Bel(\Psi - \varphi \wedge \psi)$, then contracting $\varphi \wedge \psi$ from ψ will at least reject the same beliefs as contracting φ from Ψ. This strongly relates to **(AGMes-6)**, since $Bel(\Psi - \varphi \wedge \psi) \not\models \varphi$ reveals that the contraction rejects the beliefs that are related to φ.

Lastly, we introduce the revision for epistemic states, by means of the generalized AGM revision postulates as presented by Darwiche and Pearl in [DP97], where Ψ, Φ are epistemic states, $\varphi, \psi \in \mathcal{L}$ are formulas, and $*$ is a belief change operator:

(AGMes∗1) $Bel(\Psi * \varphi) \models \varphi$

(AGMes∗2) If $Bel(\Psi) \cup \{\varphi\} \not\equiv \perp$, then $Bel(\Psi * \varphi) \equiv Bel(\Psi) \cup \{\varphi\}$

(AGMes∗3) If $\varphi \not\equiv \perp$, then $Bel(\Psi * \varphi) \not\equiv \perp$

(AGMes∗4) If $\Psi = \Phi$ and $\varphi \equiv \psi$, then $Bel(\Psi * \varphi) \equiv Bel(\Phi * \psi)$

(AGMes∗5) $Bel(\Psi * \varphi) \cup \{\psi\} \models Bel(\Psi * \varphi \wedge \psi)$

(AGMes∗6) If $Bel(\Psi * \varphi) \cup \{\psi\} \not\equiv \perp$, then $Bel(\Psi * \varphi \wedge \psi) \models Bel(\Psi * \varphi) \cup \{\psi\}$

A revision generally states the incorporation of new formulas into our present beliefs. However, in contrast to the expansion, the revision guarantees the posterior beliefs to be consistent, by rejecting those beliefs contradicting the newly added information. Therefore, the success postulate **(AGMes∗1)** states that after revising Ψ with φ, we are able to infer φ. In case that φ does not contradict the prior beliefs, it can just be added without rejecting any of the prior beliefs **(AGMes∗2)**. However, the success of a revision just holds, if the formula φ we revise the epistemic state Ψ with is not a contradiction by itself **(AGMes∗3)**. Just as the contraction for epistemic states, the revision of a formula φ is independent of its syntactic structure. Thus, when revising two equal epistemic states Ψ, Φ with equivalent formulas φ, ψ, then the resulting beliefs will also be equivalent **(AGMes∗4)**. Note that the property described by **(AGMes∗4)** does not exactly capture the same idea as the postulate originally stated by AGM. For this, it would be necessary that the beliefs of Ψ and Φ are assumed to be equivalent, instead Ψ and Φ being equal. Since the result of revising an epistemic state with a formula not only depends on its beliefs, but also on further properties, the requirement of the posterior beliefs being equivalent, if the prior beliefs are equivalent as well, is too strict. Therefore, Darwiche and Pearl weakened the antecedence of this postulate such that the epistemic states are assumed to be equal instead. **(AGMes∗5)** states that the beliefs after the incorporation of multiple information φ, ψ, represented by the conjunction $\varphi \wedge \psi$, are bounded by those of simply adding ψ after the revision with φ. Due to the unification with ψ, we know that the resulting beliefs are capable of inferring all consequences of ψ as well. The revision with $\varphi \wedge \psi$ cannot yield conclusion beyond the consequences of ψ. Thus, **(AGMes∗5)** also states that the revision with multiple information is always bounded by the formulas an expansion would add to the prior beliefs. Lastly, **(AGMes∗6)** states that if a formula ψ is consistent with the posterior beliefs of a revision $\Psi * \varphi$, then revising with $\varphi \wedge \psi$ contains the beliefs, we would obtain when simply adding ψ to our beliefs.

As can be seen from the description of the postulates above, there exist certain connections between those three fundamental belief change operations. In AGM theory, the most important connections are stated as identities, which can also be used to define an expansion, contraction or revision operator, respectively. In total, there are three identities, namely

(Levi identity) $K * \varphi = (K - \neg\varphi) + \varphi,$

(Harper identity) $K - \varphi = K \cap (K * \neg\varphi),$

(Third identity) $K + \varphi = \begin{cases} K * \varphi, & \text{if } \neg\varphi \notin K \\ \mathcal{L}, & \text{otherwise} \end{cases},$

where $+, -$ and $*$ are assumed to be an expansion, contraction and revisions, respectively, satisfying the corresponding postulates as originally stated by AGM. Note that in contrast to the **(Levi identity)** and **(Harper identity)**, the last of the stated identities does not have a specific name, which is why we refer to it as the **(Third identity)**. The **(Levi identity)** [Lev77] states that a revision can also be expressed by means of a contraction and an expansion. The contraction removes any contradicting beliefs, while the consecutively performed expansion actually adds φ and all its consequences to the present beliefs. The **(Harper identity)** [Har76] states that a contraction $K - \varphi$ can also be expressed by revising K with $\neg\varphi$ and intersecting the resulting beliefs with the prior. This way, φ and the minimum set of formulas from which φ can be inferred are removed, while all other prior beliefs are preserved. The **(Third identity)**, which directly concludes from the AGM revision postulates [AGM85, Gär88], states that an expansion $K + \varphi$ results in the same beliefs as an revision $K * \varphi$, if $\neg\varphi$ is not included in K, i.e. if the expansion with φ does not result in contradictory beliefs. This makes sense, since in this case the revision does not reject any of the present beliefs and simply adds φ to them. Otherwise, the expansion will result in contradictory beliefs. However, these identities only hold with respect to the postulates originally stated by AGM, and not for the generalized postulates stated above. In [KP17], Konieczny and Pérez discussed how these identities can be generalized such that they correspond to the AGM postulates for epistemic states, and stated several approaches. In this work, we will only make use of the simplest of the there presented approaches, which regards the identities with respect to the posterior beliefs. We refer to them as

(Levi equivalence) $Bel(\Psi * \varphi) \equiv Bel(\Psi - \neg\varphi) + \varphi,$

(Harper equivalence) $Bel(\Psi - \varphi) \equiv Bel(\Psi) \vee Bel(\Psi * \neg\varphi),$

(Third equivalence) $Bel(\Psi) + \varphi = \begin{cases} Bel(\Psi * \varphi), & \text{if } \Psi \not\models \neg\varphi \\ \mathcal{L}, & \text{otherwise} \end{cases},$

where $+$ is an expansion satisfying **(AGM+1)-(AGM+6)**, while $-$ is a contraction satisfying **(AGMes-1)-(AGMes-7)**, and $*$ is a revision satisfying **(AGMes∗1)-(AGMes∗6)**. In the following, we want to note a few things about the above-stated equivalences. First of all, we do not refer to those generalized forms as identities, because other than the identities, the equivalences cannot be used to define an expansion, contraction or revision operator. Thus, they do not fully capture the relations between the operators as originally stated by the identities. Second, the operators $*$ and $-$ are belief change operators for epistemic states and are assumed to satisfy the generalized postulates **(AGMes∗1)-(AGMes∗6)** and **(AGMes-1)-(AGMes-7)**, while the expansion $+$ still argues about belief sets only, and therefore is assumed to satisfy **(AGM+1)-(AGM+6)**. Even though, Konieczny and Pérez presented even more elaborated generalizations of the identities, that are able to capture the stated relations more accurately, the equivalences stated above are sufficient for the examinations in this work.

Finally, we want to present further revision postulates for epistemic states that form an extension of the hitherto revision postulates **(AGMes∗1)-(AGMes∗6)**. Darwiche and Pearl generalized the AGM revision postulates to epistemic states in [DP97], and argued that they are not sufficient to state properties of iterated belief revision. This is of particular interest when revising epistemic states instead of belief sets, because revising an epistemic state not only affects the corresponding beliefs, but also further properties. Therefore, Darwiche and Pearl stated four more postulates in addition to the generalized AGM revision postulates, in order to capture the notions of iterated belief revision as well. We refer to the postulates of iterated belief revision as **(DP1)-(DP4)**, where Ψ is an epistemic state and $\varphi, \psi \in \mathcal{L}$ are formulas:

(DP1) If $\varphi \models \psi$, then $Bel((\Psi * \psi) * \varphi) \equiv Bel(\Psi * \varphi)$

(DP2) If $\varphi \models \neg\psi$, then $Bel((\Psi * \psi) * \varphi) \equiv Bel(\Psi * \varphi)$

(DP3) If $Bel(\Psi * \varphi) \models \psi$, then $Bel((\Psi * \psi) * \varphi) \models \psi$

(DP4) If $Bel(\Psi * \varphi) \not\models \neg\psi$, then $Bel((\Psi * \psi) * \varphi) \not\models \neg\psi$

(DP1) states that when consecutively revising with two formulas ψ and φ, where φ is more specific than ψ, the revision with the more general information ψ is negligible, since its effect on the prior beliefs is also included in the revision with φ. **(DP2)** states that consecutively revising with two contradictory formulas φ and ψ, i.e. $\varphi \models \neg\psi$, the second revision revokes the effects of the first revision completely. **(DP3)** states that when a certain information ψ is believed after revising Ψ with φ, then explicitly incorporating ψ into our beliefs before revising them with φ does not change that ψ is believed afterwards. Lastly, **(DP4)** states that if our beliefs do not contradict ψ after a revision with φ, then they especially do not contradict ψ, when we explicitly incorporate it into our beliefs before revising them with φ. These properties are also said to formalize the principle of conditional preservation for belief revisions, which in contrast to the minimum change paradigm states that conditional relations in our epistemic state are preserved, if there is no reason to reject them. Darwiche and Pearl also discussed in [DP97] that the minimum change paradigm and the principle of conditional preservation are not always compatible, since the preservation of conditional relations often induces additional propositional changes, and vice-versa.

2.4 Ordinal Conditional Functions

In epistemology, terms like truth, knowledge, belief and plausibility can be separated into two categories – determinism and probabilism. This dichotomy does not characterize a strict division and allows epistemic terms to be discussed deterministically as well as probabilistically. The different notions do not necessarily have to exclude each other. More often, the opposite is the case in which the different notions benefit from each other and together deduce more profound insights. In probability theory, it was possible to argue about certain terms, which could not be discussed deterministically for a long time, including the irrelevance of information and the change in certain beliefs with respect to a given observation or fact. A major reason why arguing about those terms deterministically was impossible for such a long time is the fact that deterministic approaches were not able to handle uncertain knowledge. In contrast to secure knowledge, uncertain knowledge is not simply true or false, but allows exceptions to exist, e.g. we know that birds generally fly, but we also know that there exists birds to which this rule does not apply. Thus, uncertain knowledge can be regarded as plausible rules rather than strict facts. While this kind of uncertainty can be inherently described by probability theory, a deterministic representation is more complicated, since exceptions as described above were not intended in classical logic. Motivated by the capabilities of handling uncertain and conditional knowledge in probability theory, Spohn developed a qualitative abstraction of discrete probability distributions to argue about the above-mentioned terms logically. This abstraction is know as *ordinal conditional functions* (OCFs) [Spo88] and forms a common way of representing epistemic states capable of handling uncertain knowledge.

In the following, we will elaborate the fundamental definitions and properties of OCFs as needed in this work, which includes among others, the belief in and plausibility of propositional and conditional knowledge, and the importance of minimal models.

Ordinal conditional functions as epistemic states. In the further course, we want to discuss OCFs as epistemic states. Thereby, we elaborate how OCFs can be used to represent knowledge and how their beliefs can be obtained. Furthermore, we state the basic properties and equivalences of OCFs as needed in this work. Afterwards, we show that OCFs follow the concept of faithful assignments stated in Section 2.3, which is a common assumption when working with general epistemic states. We start with the definition of OCFs (Def. 2.29), since they form the basis for everything discussed further in this section.

Definition 2.29. [Spo88] *An* ordinal conditional function (OCF) $\kappa : \Omega_\Sigma \to \mathbb{N}_0 \cup \{\infty\}$ *over signature* Σ *assigns a rank to each interpretation* $\omega \in \Omega_\Sigma$ *with* $\kappa^{-1}(0) \neq \emptyset$.

As already mentioned above, OCFs are epistemic states that can be understood as a qualitative abstraction of probability functions and enrich the beliefs of our epistemic states by a plausibility ranking of interpretations, which can be compared to a probability function assigning probabilities to possible outcomes. But in contrast to probabilities, an interpretation appears more plausible the lower its rank is. Thus, the rank an OCF assigns to an interpretation can be understood as a degree of disbelief. The interpretations that are assigned to rank 0 are therefore considered the most plausible. If an interpretation is assigned to rank ∞, it means that it is not just unlikely, but completely excluded. Thus, assigning an interpretation to rank ∞ corresponds to the idea of something being impossible, and therefore to a probability of 0. However, in this work we only consider OCFs that do not assign interpretations to rank ∞, since it only makes arguing about them more technically without providing any advantages. From a cognitive perspective, one can also argue that excluding certain interpretations completely is not desirable, since one can usually not exclude certain considerations no matter how unlikely they seem. This is closely related to the question what truth actually is, and that even knowledge we consider as facts is based on assumption that theoretically can be disproved. The additional condition that the inverse image $\kappa^{-1}(0)$ must not be empty, states that there must always exist most plausible interpretations. In the further course, we refer to the most plausible interpretations of κ as $[\![\kappa]\!]$ (Def. 2.30).

Definition 2.30. [Spo88] *Let κ be an OCF over signature Σ. The* most plausible interpretations *of κ are given by $[\![\kappa]\!] = \{\omega \in \Omega_\Sigma \mid \kappa(\omega) = 0\}$.*

Since epistemic states can generally be used to describe our beliefs about the objects and concepts of our world, represented by the signature elements Σ, we further want to define, in which case a proposition $\varphi \in \mathcal{L}_\Sigma$ is believed by an OCF κ. For this, we make use of the plausibility ranking as already introduced for interpretations, and first expand it to formulas in Def. 2.31.

Definition 2.31. [Spo88] *Let κ be an OCF over signature Σ, then the* rank of a formula *$\varphi \in \mathcal{L}_\Sigma$ is defined as*

$$\kappa(\varphi) = \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma \textit{ and } \omega \models \varphi\}.$$

Thereby, the rank of a formula φ is determined by the minimum rank among the interpretations satisfying φ. This means that a formula is always as plausible as its most plausible model. When we now want to know, if a certain formula φ is believed by κ, we simply compare whether φ or $\neg\varphi$ appears more plausible. We say that φ is believed by κ, denoted as $\kappa \models \varphi$, if and only if φ is more plausible than $\neg\varphi$ (Def. 2.32).

Definition 2.32. [Spo88] *Let κ be an OCF over Σ and $\varphi \in \mathcal{L}_\Sigma$ a formula.*

$$\kappa \models \varphi \Leftrightarrow \kappa(\varphi) < \kappa(\neg\varphi)$$

In case that φ and $\neg\varphi$ are assigned to the same rank, neither of them can be concluded, since this would mean that we believe in contradicting propositions. On the contrary, we know that φ cannot be inferred by κ, if and only if $\neg\varphi$ is assigned to rank 0 (Lem. 2.33).

Lemma 2.33. [Spo88] *Let κ be an OCF over signature Σ, then for each formula $\varphi \in \mathcal{L}_\Sigma$ the following holds:*

$$\kappa \not\models \varphi \Leftrightarrow \kappa(\neg\varphi) = 0$$

Since we now know that a proposition is believed when it appears more plausible than its negation, we can further describe the belief set $Bel(\kappa)$ of an OCF κ as the set of all formulas that can be inferred it (Def. 2.34). Note that $Bel(\kappa)$ must be deductively closed, since it already contains all formulas that can be inferred from κ.

Definition 2.34. [Spo88] *Let κ be an OCF over Σ. The* belief set *of κ*

$$Bel(\kappa) = \{\varphi \in \mathcal{L}_\Sigma \mid \kappa \models \varphi\}$$

is the deductively closed set formulas that can be inferred by κ.

In the following, we complete the overall picture by stating two more basic properties of OCFs that directly conclude from their definition. For each formula $\varphi \in \mathcal{L}_\Sigma$, we know that φ, $\neg\varphi$ or both must be assigned to rank 0, since each interpretation $\omega \in \Omega_\Sigma$ is either a model of φ or $\neg\varphi$. This can be traced back to the condition that there must exist interpretations with rank 0.

Proposition 2.35. [Spo88] *Let κ be an OCF over signature Σ, then $\kappa(\varphi) = 0$ or $\kappa(\neg\varphi) = 0$ holds for each formula $\varphi \in \mathcal{L}_\Sigma$.*

Prop. 2.35 also illustrates that it is necessary, but not sufficient for a formula to have rank 0 in order to be believed by an OCF. Furthermore, we can conclude that the rank of a disjunction $\varphi \vee \psi$ must be the minimum rank of either φ or ψ (Lem. 2.36). This holds, since all models of φ and ψ are models of $\varphi \vee \psi$ as well.

Lemma 2.36. [Spo88] *Let κ be an OCF over signature Σ, then for all formulas $\varphi, \psi \in \mathcal{L}_\Sigma$ the following holds:*

$$\kappa(\varphi \vee \psi) = \min\{\kappa(\varphi), \kappa(\psi)\}$$

Given the basic definitions and properties of OCFs, we further want to state some equivalences regarding its most plausible interpretations and beliefs, which will be needed for the further examinations in this work. First, we state in Prop. 2.37 that an interpretation ω is assigned to rank 0, if and only if its included in $[\![\kappa]\!]$, and furthermore we know that it then must be a model of $Bel(\kappa)$ as well.

Proposition 2.37. *Let κ be an OCF over signature Σ and $\omega \in \Omega_\Sigma$ an interpretation, then the following relations are equivalent:*

1. $\kappa(\omega) = 0$
2. $\omega \in [\![\kappa]\!]$
3. $\omega \models Bel(\kappa)$

From this, we can further conclude the equivalences given in Prop. 2.38, which state that the beliefs of two OCF κ, κ' are equivalent, if and only if their most plausible interpretations are equal. Moreover, this can also be expressed by means of the corresponding theories.

Proposition 2.38. *Let κ, κ' be OCFs over the same signature Σ, then the following relations are equivalent:*

1. $Bel(\kappa) \equiv Bel(\kappa')$
2. $[\![\kappa]\!] = [\![\kappa']\!]$
3. $Th([\![\kappa]\!]) \equiv Th([\![\kappa']\!])$

From Prop. 2.38, we can conclude that the beliefs of an OCF can especially be stated as the theory of its most plausible interpretations (Lem. 2.39). Thus, the models of an belief set exactly correspond to the most plausible interpretations (Lem. 2.40).

Lemma 2.39. *Let κ be an OCF, then $Bel(\kappa) \equiv Th([\![\kappa]\!])$ holds.*

Lemma 2.40. *Let κ be an OCF, then $[\![Bel(\kappa)]\!] = [\![\kappa]\!]$ holds.*

Analogously to Prop. 2.38, we can state that the beliefs of an OCF κ infer those of another OCF κ', if and only if $[\![\kappa]\!]$ is included in $[\![\kappa']\!]$ respectively (Prop. 2.41). Again, the same holds for the corresponding theories.

Proposition 2.41. *Let κ, κ' be OCFs over the same signature Σ, then the following relations are equivalent:*

1. $Bel(\kappa) \models Bel(\kappa')$
2. $\kappa \models Bel(\kappa')$
3. $[\![\kappa]\!] \subseteq [\![\kappa']\!]$
4. $Th([\![\kappa]\!]) \models Th([\![\kappa']\!])$

After presenting the basic definitions and properties of OCFs as needed in this work, we want to elaborate the relation of OCFs and faithful assignments as mentioned in Section 2.3. There, we described faithful assignments as a commonly assumed property of epistemic states, since they imply important belief change properties. Next, we show that the order of the interpretations induced by an OCF are obtained faithfully. For this, we first like to specify this order in Prop. 2.42.

Proposition 2.42. [KI01] *Let κ be an OCF over signature Σ, then κ induces a total preorder $\preceq_\kappa$ with*

$$\omega \preceq_\kappa \omega' \Leftrightarrow \kappa(\omega) \leq \kappa(\omega'),$$

for all interpretations $\omega, \omega' \in \Omega_\Sigma$.

This induced total preorder follows a faithful assignment (Def. 2.27), since all of the necessary conditions are satisfied (Prop. 2.43). Therefore, we can make use of OCFs in the further course, whenever an epistemic state with a faithful ranking is assumed.

Proposition 2.43. *Let κ be an OCF over signature Σ. The corresponding induced total preorder $\preceq_\kappa$ satisfies*

1. $\omega, \omega' \models Bel(\kappa)$, *only if* $\omega =_\kappa \omega'$,
2. $\omega \models Bel(\kappa)$ *and* $\omega' \not\models Bel(\kappa)$, *only if* $\omega \prec_\kappa \omega'$,
3. $\kappa = \kappa'$, *only if* $\preceq_\kappa = \preceq_{\kappa'}$,

where $\omega =_\kappa \omega'$ *is defined as* $\omega \preceq_\kappa \omega'$ *and* $\omega' \preceq_\kappa \omega$; *and* $\omega \prec_\kappa \omega'$ *is defined as* $\omega \preceq_\kappa \omega'$ *and* $\omega' \not\preceq_\kappa \omega$. *Therefore* $\kappa \mapsto \preceq_\kappa$ *is a faithful assignment.*

Proof of Prop. 2.43.

1. $\omega, \omega' \models Bel(\kappa)$, only if $\omega =_\kappa \omega'$:

$$\begin{aligned} \omega, \omega' \models Bel(\kappa) &\Leftrightarrow \kappa(\omega) = 0 = \kappa(\omega') && \text{(Prop. 2.37)} \\ &\Rightarrow \kappa(\omega) \leq \kappa(\omega') \text{ and } \kappa(\omega') \leq \kappa(\omega) \\ &\Leftrightarrow \omega \preceq_\kappa \omega' \text{ and } \omega' \preceq_\kappa \omega && \text{(Prop. 2.42)} \\ &\Leftrightarrow \omega =_\kappa \omega' \end{aligned}$$

2. $\omega \models Bel(\kappa)$ and $\omega' \not\models Bel(\kappa)$, only if $\omega \prec_\kappa \omega'$:

$$\begin{aligned} &\omega \models Bel(\kappa) \text{ and } \omega' \not\models Bel(\kappa) \\ \Leftrightarrow\ &\kappa(\omega) = 0 \text{ and } \kappa(\omega') > 0 && \text{(Prop. 2.37)} \\ \Leftrightarrow\ &\omega \preceq_\kappa \omega' \text{ and } \omega' \not\preceq_\kappa \omega && \text{(Prop. 2.42)} \\ \Leftrightarrow\ &\omega \prec_\kappa \omega' \end{aligned}$$

   Since, we showed the equivalence of $\omega \models Bel(\kappa), \omega' \not\models Bel(\kappa)$ and $\omega \prec_\kappa \omega'$, we know that the implication stated in (2.) holds as well.

3. $\kappa = \kappa'$, only if $\preceq_\kappa = \preceq_{\kappa'}$: We prove (3.) by showing that if the total preorders $\preceq_\kappa$, $\preceq_{\kappa'}$ are not equal, then κ and κ' cannot be equal either.

$$\begin{aligned} &\preceq_\kappa \neq \preceq_{\kappa'} \\ \Leftrightarrow\ &\text{there exist } \omega, \omega' \in \Omega_\Sigma \text{ with} \\ &(\omega \preceq_\kappa \omega' \text{ and } \omega \not\preceq_{\kappa'} \omega') \text{ or } (\omega \not\preceq_\kappa \omega' \text{ and } \omega \preceq_{\kappa'} \omega') \\ \Leftrightarrow\ &\text{there exist } \omega, \omega' \in \Omega_\Sigma \text{ with} \\ &(\kappa(\omega) \leq \kappa(\omega') \text{ and } \kappa'(\omega) \not\leq \kappa'(\omega')) && \text{(Prop. 2.37)} \\ &\ \text{or } (\kappa(\omega) \not\leq \kappa(\omega') \text{ and } \kappa'(\omega) \leq \kappa'(\omega')) \\ \Rightarrow\ &\text{there exist } \omega \in \Omega_\Sigma \text{ with } \kappa(\omega) \neq \kappa'(\omega) \\ \Leftrightarrow\ &\kappa \neq \kappa' \end{aligned}$$

□

Handling uncertainties with ordinal conditional functions. One of the key strengths of OCFs is their capability of handling uncertain and conditional knowledge, which again originates from the fact that OCFs can be understood as a qualitative abstraction of probability functions, for which we know that they are capable of handling both of the above as well. Even if we mainly focus on propositional knowledge in this work, we want to briefly introduce how OCFs can be used to represent conditional, and therefore uncertain knowledge in the following. For this, we will introduce conditionals, which can be understood as plausible rules, their corresponding three-valued satisfaction relation, and furthermore under which circumstances they are believed by an OCF.

First, we intuitively recall conditional probability functions and show why the semantics of classical propositional logic are not sufficient for representing uncertain knowledge. A conditional probability function $P(X = x|Y = y)$ describes the probability that X takes on the value x, when we assume or already know that $Y = y$ holds. So, with $Y = y$ given, the probabilities of X taking on a certain value change accordingly. If $P(X = x|Y = y) = 1$ would hold, we would know that given $Y = y$, X must definitely take on the value x, since all other values that might be possible are assigned to a probability of 0. In this case, the relation of $X = x$ and $Y = y$ can be described as a strict rule: if $Y = y$, then $X = x$. Such strict rules can also be described in classical logic by means of a material implication $X \to Y$. However, the probability mass will generally not collapse into a single value like this, but will rather redistribute over all values X can take on. Therefore, $P(X = x|Y = y)$ describes how plausible $X = x$ seems under the assumption $Y = y$. This means, there is an uncertainty about the behaviour on which value X takes on, and therefore strict rules such as the material implication are not capable of capturing this phenomena in general. To finally argue about this kind of conditional knowledge, we need if-then rules capable of abstracting the uncertainty of conditional probabilities. This leads us to the definition of conditionals (Def. 2.44).

Definition 2.44. [Spo88] *The* set of conditionals $(\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ *over a language* $\mathcal{L}_\Sigma$ *is defined as* $(\mathcal{L}_\Sigma|\mathcal{L}_\Sigma) = \{(\psi|\varphi) \mid \varphi, \psi \in \mathcal{L}_\Sigma\}$.

Inspired by the notation of conditional probabilities, we denote a conditional by $(\psi|\varphi)$, where φ is the antecedence and ψ the conclusion. $(\psi|\varphi)$ can be read as ψ *follows plausibly from* φ or φ *usually implies* ψ, and therefore can be considered as a plausible rule. Even though we understand conditionals this way intuitively, they are of a purely syntactic nature. In order to capture the uncertainty of plausible rules semantically, it is not sufficient to use a binary satisfaction relation as in classical logic, because this way a rule could only be *true* or *false*. Instead, we define a ternary satisfaction relation (Def. 2.45), in which we distinguish between the *verification*, *falsification* and *satisfaction* of conditionals. This definition goes back to de Finetti who considered conditionals as general indicator functions [DeF74].

Definition 2.45. [KI01] *Let* $(\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ *be a set of conditionals over the language* $\mathcal{L}_\Sigma$ *and* Ω_Σ *the corresponding set of interpretations. For each conditional* $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ *and* $\omega \in \Omega_\Sigma$*, the following three-valued satisfaction relation holds:*

$$\omega \text{ falsifies } (\psi|\varphi) \Leftrightarrow \omega \models \varphi \wedge \neg\psi$$

$$\omega \text{ satisifies } (\psi|\varphi) \Leftrightarrow \omega \models \varphi \Rightarrow \psi$$
$$\omega \text{ verifies } (\psi|\varphi) \Leftrightarrow \omega \models \varphi \wedge \psi$$

An interpretation *falsifies* a conditional, when the conclusion holds but the antecedence does not. In this case, the interpretation disagrees with the plausible rule. A conditional is *verified* by an interpretations, if both the antecedence and the conclusion hold. Finally, an interpretation ω can satisfy a conditional $(\psi|\varphi)$ in two cases, in which ω either verifies $(\psi|\varphi)$ or is not applicable at all, since it contradicts the antecedence of the plausible rule, namely φ. In simple terms, a conditional is satisfied by an interpretation if it is not falsified. Given the three-valued satisfaction relation above, we define the rank of a conditional in Def. 2.46.

Definition 2.46. [KI01] *Let κ be and OCF over signature Σ and $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ a conditional.*

$$\kappa(\psi|\varphi) = \kappa(\varphi \wedge \psi) - \kappa(\varphi)$$

Determining the rank of a conditional $(\psi|\varphi)$, can be viewed as determining the rank of $\varphi \wedge \psi$ in an OCF κ that only argues about the models of φ. These are the interpretations that either falsify or verify $(\psi|\varphi)$. Conditionalizing κ to the models of φ corresponds to the assumption that φ must hold according to the antecedence of $(\psi|\varphi)$. In Def. 2.46, this corresponds to subtracting $\kappa(\varphi)$, which maintains the condition $\kappa^{-1}(0) \neq \emptyset$ when removing the models of $\neg\varphi$. In this conditionalized OCF, we then determine the rank of $\varphi \wedge \psi$, which corresponds to the minimum rank among those interpretations verifying $(\psi|\varphi)$. Furthermore, we can define whether an OCF infers a conditional (Def. 2.47) analogously to the inference of propositions stated in Def. 2.32.

Definition 2.47. [KI01] *Let κ be an OCF over Σ and $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ a conditional.*

$$\kappa \models (\psi|\varphi) \Leftrightarrow \kappa(\psi|\varphi) < \kappa(\neg\psi|\varphi)$$

This means that an OCF believes a conditional, if and only if its verification seems more plausible than its falsification. Similarly, we say that a conditional $(\psi|\varphi)$ is not believed by an OCF, if and only if $(\neg\psi|\varphi)$ is at least as plausible as $(\psi|\varphi)$ (Lem. 2.48).

Lemma 2.48. *Let κ be an OCF over Σ and $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ a conditional.*

$$\kappa \not\models (\psi|\varphi) \Leftrightarrow \kappa(\neg\psi|\varphi) \leq \kappa(\psi|\varphi)$$

Finally, in Prop. 2.49 we state some equivalences regarding the inference of a conditional that will be useful for further examinations, and finish this paragraph on OCFs and uncertain knowledge by giving an example that summarizes the previously discussed properties of OCFs regarding conditionals (Ex. 2.2).

Proposition 2.49. *Let κ be an OCF over signature Σ and $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ a conditional, then the following relations are equivalent:*

1. $\kappa \models (\psi|\varphi)$

2. $\kappa(\psi|\varphi) < \kappa(\neg\psi|\varphi)$
3. $\kappa(\varphi \wedge \psi) < \kappa(\varphi \wedge \neg\psi)$

Example 2.2. *In this example, we want to illustrate how OCFs can be used to handle uncertain knowledge represented by conditionals. For this, we consider the OCF κ as given in Tab. 4 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	pbf
2	$\overline{p}b\overline{f}$, $pb\overline{f}$
1	$\overline{p}bf$, $p\overline{b}f$
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}\,\overline{b}f$, $p\overline{b}\,\overline{f}$

$\kappa_{(\cdot|b)}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	pbf
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}bf$

Table 4: Left: OCF κ over signature Σ_{Tweety}. Right: OCF κ over signature Σ_{Tweety}, but restricted to the models of b.

According to Def. 2.45, we know that a conditional can be inferred by an OCF, if and only if its verification seems more plausible than its falsification. Considering the conditional $(f|b)$, which reads birds usually fly, *we know that κ infers $(f|b)$, if and only if:*

$$
\begin{aligned}
& \kappa \models (f|b) && \\
\Leftrightarrow\ & \kappa(f|b) < \kappa(\neg f|b) && \text{(Def. 2.45)} \\
\Leftrightarrow\ & \kappa(f \wedge b) - \kappa(b) < \kappa(\neg f \wedge b) - \kappa(b) && \text{(Def. 2.46)} \\
\Leftrightarrow\ & 1 - 1 < 2 - 1 && \\
\Leftrightarrow\ & 0 < 1 && \checkmark
\end{aligned}
$$

Thus, we know that birds usually fly *is believed by κ.*

As already mentioned above, the subtraction of $\kappa(b)$ can be viewed as restricting κ to the models of the antecedence b. Thus, $\kappa(f|b)$ corresponds to determining the rank of f under the assumption that b holds. The OCF that would result from actually restricting κ to the models of b is also illustrated in Tab. 4. We can see that the ranks assigned to $f \wedge b$ and $\neg f \wedge b$ by $\kappa_{(\cdot|b)}$ correspond to the ranks of $(f|b)$ and $(\neg f|b)$ in κ, and thus $\kappa_{(\cdot|b)}$ infers $(f|b)$ as well:

$$
\begin{aligned}
& \kappa_{(\cdot|b)} \models (f|b) && \\
\Leftrightarrow\ & \kappa_{(\cdot|b)}(f|b) < \kappa_{(\cdot|b)}(\neg f|b) && \text{(Def. 2.45)} \\
\Leftrightarrow\ & \kappa_{(\cdot|b)}(f \wedge b) - \kappa_{(\cdot|b)}(b) < \kappa_{(\cdot|b)}(\neg f \wedge b) - \kappa_{(\cdot|b)}(b) && \text{(Def. 2.46)} \\
\Leftrightarrow\ & 0 - 0 < 1 - 0 && \\
\Leftrightarrow\ & 0 < 1 && \checkmark
\end{aligned}
$$

Minimal models and refinements of OCFs. Lastly, we want to argue about the concept of minimal models, which will be essential for this work. Thus, we state their definition and fundamental properties as needed in this work, and furthermore the refinement relation of OCFs, which is strongly related to them.

First, we state the definition of minimal models in Def. 2.50. Thereby, minimal models define the models of a formulas that are most plausible according to the total preorder $\preceq_\kappa$ induced by an OCF κ.

Definition 2.50. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$, and $\varphi \in \mathcal{L}_\Sigma$ a formula, then the* minimal models of φ *are defined as*

$$\min\{[\![\varphi]\!], \preceq_\kappa\} = \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = \kappa(\varphi)\}.$$

Note that the minimality of the selected models is implicitly stated by the equality $\kappa(\omega) = \kappa(\varphi)$, since the rank of φ is already given by its most plausible models. Therefore, by assuming $\kappa(\omega) = \kappa(\varphi)$, we also assume that there does not exist other models of φ with a lower rank than ω.

In the following, we state several properties that are necessary when arguing about minimal models. First, we state that if we assume that the minimal models of φ and ψ are not disjunct, then both formulas must be assigned to the same rank (Lem. 2.51).

Lemma 2.51. *Let κ be an OCF over signature Σ and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas.*

$$\textit{If } \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} \neq \emptyset, \textit{ then } \kappa(\varphi) = \kappa(\psi)$$

This property is straightforward, since we know that all minimal models of φ are assigned to a certain rank r, while all minimal models of ψ are assigned to a certain rank r'. Obviously, φ and ψ can only share minimal models if they are assigned to the same rank $r = r'$.

From Lem. 2.36, we know that the rank of a disjunction $\varphi \vee \psi$ is given by the minimum rank of φ and ψ. Considering the minimal models of $\varphi \vee \psi$, we can make use of Lem. 2.36 and show that they equal the unification of the minimal models of φ and ψ, if and only if φ and ψ are assigned to the same rank (Prop. 2.52)

Proposition 2.52. *Let κ be an OCF over signature Σ and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas.*

$$\min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\} \Leftrightarrow \kappa(\varphi) = \kappa(\psi)$$

Proof of Prop. 2.52. We prove the equivalence stated in Prop. 2.52, by showing that $\min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\}$ implies that φ and ψ are assigned to the same rank, and that if φ and ψ are assigned to the same rank, then $\min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\}$ holds. In the following, we refer to these two cases as ($\Rightarrow$) and ($\Leftarrow$)

Case ($\Rightarrow$):
By Def. 2.50, we know $\kappa(\omega) = \kappa(\varphi \vee \psi)$ holds for each $\omega \in \min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\}$. Further we know $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} \neq \emptyset$, since we assumed $\min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq$

$\min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\}$, and therefore we can conclude $\kappa(\varphi) = \kappa(\varphi \vee \psi)$ due to Lem. 2.51. Thus, $\kappa(\omega) = \kappa(\varphi \vee \psi)$ holds for each $\omega \in \min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\}$. The same holds analogously for the minimal models of ψ. Thus, we can conclude that $\kappa(\varphi) = \kappa(\psi)$ holds, if we assume $\min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\} = \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cup \min\{\llbracket\psi\rrbracket, \preceq_\kappa\}$.

Case ($\Leftarrow$):
In the further course, we assume $\kappa(\varphi) = \kappa(\psi)$ and show that under this assumption $\min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\} = \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cup \min\{\llbracket\psi\rrbracket, \preceq_\kappa\}$ holds.

$$\begin{aligned}
&\min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\} \\
&= \{\omega \in \llbracket\varphi \vee \psi\rrbracket \mid \kappa(\omega) = \kappa(\varphi \vee \psi)\} && \text{(Def. 2.50)} \\
&= \{\omega \in \llbracket\varphi\rrbracket \cup \llbracket\psi\rrbracket \mid \kappa(\omega) = \kappa(\varphi \vee \psi)\} && \text{(Lem. 2.10)} \\
&= \{\omega \in \llbracket\varphi\rrbracket \mid \kappa(\omega) = \kappa(\varphi \vee \psi)\} \cup \{\omega \in \llbracket\psi\rrbracket \mid \kappa(\omega) = \kappa(\varphi \vee \psi)\} \\
&= \{\omega \in \llbracket\varphi\rrbracket \mid \kappa(\omega) = \min\{\kappa(\varphi), \kappa(\psi)\}\} \\
&\quad \cup \{\omega \in \llbracket\psi\rrbracket \mid \kappa(\omega) = \min\{\kappa(\varphi), \kappa(\psi)\}\} && \text{(Lem. 2.36)} \\
&= \{\omega \in \llbracket\varphi\rrbracket \mid \kappa(\omega) = \kappa(\varphi)\} \cup \{\omega \in \llbracket\psi\rrbracket \mid \kappa(\omega) = \kappa(\psi)\} && (\kappa(\varphi) = \kappa(\psi)) \\
&= \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cup \min\{\llbracket\psi\rrbracket, \preceq_\kappa\} && \text{(Def. 2.50)}
\end{aligned}$$

□

Given Lem. 2.36 and Prop. 2.52, we can further express the minimal models of $\varphi \vee \psi$ as stated in Lem. 2.53.

Lemma 2.53. *Let κ be an OCF over signature Σ and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas.*

$$\min\{\llbracket\varphi \vee \psi\rrbracket, \preceq_\kappa\} = \begin{cases} \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\}, & \text{if } \kappa(\varphi) < \kappa(\psi) \\ \min\{\llbracket\psi\rrbracket, \preceq_\kappa\}, & \text{if } \kappa(\psi) < \kappa(\varphi) \\ \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cup \min\{\llbracket\psi\rrbracket, \preceq_\kappa\}, & \text{otherwise} \end{cases}$$

Next, we state some properties regarding the relations of disjunctions $\varphi \vee \psi$ and conjunctions $\varphi \wedge \psi$ to the minimal models of φ and ψ. In Prop. 2.54 we state that the intersection of the minimal models of φ and ψ is always included in the minimal models of their disjunction and conjunction, respectively.

Proposition 2.54. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$ and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, then the following subset relations hold:*

$$\min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cap \min\{\llbracket\psi\rrbracket, \preceq_\kappa\} \subseteq \min\{\llbracket\psi \wedge \varphi\rrbracket, \preceq_\kappa\}$$
$$\min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cap \min\{\llbracket\psi\rrbracket, \preceq_\kappa\} \subseteq \min\{\llbracket\psi \vee \varphi\rrbracket, \preceq_\kappa\}$$

Proof of Prop. 2.54. We will proof the subset relation stated in Prop. 2.54 for the conjunction $\psi \wedge \varphi$ first, and for the disjunction $\psi \vee \varphi$ afterwards.

Proof of $\min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cap \min\{\llbracket\psi\rrbracket, \preceq_\kappa\} \subseteq \min\{\llbracket\psi \wedge \varphi\rrbracket, \preceq_\kappa\}$:
In the following we distinguish two cases. In the first case ($= \emptyset$) we show that the subset relation holds if $\min\{\llbracket\varphi\rrbracket, \preceq_\kappa\} \cap \min\{\llbracket\psi\rrbracket, \preceq_\kappa\} = \emptyset$. In the second case ($\neq \emptyset$)

we show that it also holds if the intersection is not empty.

Case $(= \emptyset)$*:* In this case the subset relation holds trivially, since $\emptyset \subseteq \Theta$ holds for every arbitrary set Θ.

Case $(\neq \emptyset)$*:* For each of interpretation $\omega \in \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\}$, we know that $\kappa(\varphi) = \kappa(\omega) = \kappa(\psi)$ holds due to Lem. 2.51. Moreover, we know that each $\omega \in \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\}$ must be a model of $\varphi \wedge \psi$, because of

$$\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} \subseteq [\![\varphi]\!] \cap [\![\psi]\!] = [\![\varphi \wedge \psi]\!]. \qquad \text{(Lem. 2.10)}$$

Thus, we know that there cannot exist models of $\varphi \wedge \psi$ that are assigned to smaller ranks than φ or ψ, because otherwise there would exist a model ω satisfying φ and ψ with $\kappa(\omega) < \kappa(\varphi)$ and $\kappa(\omega) < \kappa(\psi)$. This concludes that $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\psi \wedge \varphi]\!], \preceq_\kappa\}$ must hold.

Proof of $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} \subseteq \min\{[\![\psi \vee \varphi]\!], \preceq_\kappa\}$:
We again distinguish the two cases in which $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\}$ is either assumed to be empty $(= \emptyset)$ or to contain at least a single interpretation $(\neq \emptyset)$.

Case $(= \emptyset)$*:* Just as for the first part of the proof, the subset relation holds trivially if $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} = \emptyset$.

Case $(\neq \emptyset)$*:* By assumption, we can make use of Lem. 2.51 and conclude that $\kappa(\varphi) = \kappa(\psi)$ must hold. Due to Lem. 2.36, we especially know $\kappa(\varphi) = \kappa(\varphi \vee \psi) = \kappa(\psi)$. From this we can further conclude

$$\min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\}. \qquad \text{(Lem. 2.53)}$$

Since the intersection of two sets $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_\kappa\}$ is always a subset of their unification $\min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\}$, we can finally conclude that the subset relation stated in Prop. 2.52 holds. $\square$

From Prop. 2.54 we can further derive that if we assume the intersection of the minimal models of φ and ψ to be non-empty, we know that the intersection is even equal to the minimal models of $\varphi \wedge \psi$ (Lem. 2.55). This is a special case of the property stated in Prop. 2.54.

Lemma 2.55. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$ and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas.*

$$\begin{aligned} &\textit{If } \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} \neq \emptyset \\ &\textit{then } \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\psi \wedge \varphi]\!], \preceq_\kappa\} \end{aligned}$$

Finally, we want to state the refinement relation of two OCFs κ and κ' (Def. 2.56).

Definition 2.56. *Let κ and κ' be OCFs over the same signature Σ. κ is a* refinement *of κ', denoted by $\kappa \sqsubseteq \kappa'$, if and only if for each pair of interpretations $\omega, \omega' \in \Omega_\Sigma$ it holds that if $\omega \preceq_\kappa \omega'$, then $\omega \preceq_{\kappa'} \omega'$.*

We say that κ refines κ', denoted by $\kappa \sqsubseteq \kappa'$, in case that each relation between the interpretations that holds in κ also holds in κ'. However, $\preceq_{\kappa'}$ is allowed to contain relations that are not valid in $\preceq_\kappa$. It is important to note that due to the condition that the relations of $\preceq_\kappa$ must also hold in $\preceq_{\kappa'}$, it is not possible for $\preceq_\kappa$ to invert any of the relations in $\preceq_{\kappa'}$. Metaphorically, one can imagine that a refinement of κ' can be obtained by observing it with a magnifying glass. Thus, taking a closer look might reveal that some of the interpretations that appeared equal in rank before, are actually assigned to different ranks. This way we obtained a more detailed view on κ' that does not contradict the previous. Furthermore, the refinement relation implies that the most plausible interpretations of κ must be included in those of κ', if $\kappa \sqsubseteq \kappa'$ holds (Prop. 2.57).

Proposition 2.57. *Let κ, κ' be OCFs over signature Σ, then the following holds:*

$$\textit{If } \kappa \sqsubseteq \kappa', \ \textit{then } [\![\kappa]\!] \subseteq [\![\kappa']\!].$$

Proof of Prop. 2.57. We prove Prop. 2.57 by means of a contraposition. For this, we show that κ cannot be a refinement of κ', if we assume that $[\![\kappa]\!]$ is not a subset of $[\![\kappa']\!]$:

$$\text{If } [\![\kappa]\!] \nsubseteq [\![\kappa']\!], \ \text{then } \kappa \not\sqsubseteq \kappa'.$$

This can also be expressed as

$$\begin{aligned}&\text{if there exist } \omega \in [\![\kappa]\!] \text{ with } \omega \notin [\![\kappa']\!],\\ &\text{then there exist } \omega, \omega' \in \Omega_\Sigma \text{ with } \omega \preceq_\kappa \omega' \text{ and } \omega \npreceq_{\kappa'} \omega'.\end{aligned}$$

In the further course, we choose $\omega \in [\![\kappa]\!]$ freely, such that $\omega \notin [\![\kappa']\!]$ holds. Due to the assumption $[\![\kappa]\!] \nsubseteq [\![\kappa']\!]$, we know that such ω do exist. Further, let $\omega' \in [\![\kappa']\!]$ be an arbitrary interpretation. Choosing ω and ω' as stated above, the following holds:

$$\begin{aligned}&\omega \preceq_\kappa \omega' &&\\ \Leftrightarrow\ &\kappa(\omega) \leq \kappa(\omega') && \text{(Prop. 2.42)}\\ \Leftrightarrow\ &0 \leq \kappa(\omega') && (\omega \in [\![\kappa]\!], \text{Prop. 2.37})\end{aligned}$$

Since 0 is smallest rank possible, $0 \leq \kappa(\omega')$ holds for each $\omega' \in \Omega_\Sigma$. Furthermore, we know that the following holds as well:

$$\begin{aligned}&\omega \npreceq_{\kappa'} \omega' &&\\ \Leftrightarrow\ &\kappa'(\omega) \nleq \kappa'(\omega') && \text{(Prop. 2.42)}\\ \Leftrightarrow\ &\kappa'(\omega) > \kappa'(\omega') &&\\ \Leftrightarrow\ &\kappa'(\omega) > 0 && (\omega' \in [\![\kappa']\!], \text{Prop. 2.37})\end{aligned}$$

Since $\omega \notin [\![\kappa']\!]$, we know due to Prop. 2.37 that $\kappa'(\omega) \neq 0$, and therefore that $\kappa'(\omega) > 0$ holds.

In conclusion, we showed that κ cannot be a refinement of κ', if we assume $[\![\kappa]\!] \nsubseteq [\![\kappa']\!]$. By means of contraposition, this concludes that $[\![\kappa]\!] \subseteq [\![\kappa']\!]$ must hold, if κ refines κ', i.e. $\kappa \sqsubseteq \kappa'$. $\square$

Assuming that κ refines κ' also affects the relations of the minimal models. This will be discussed in the later sections. Finally, we want to give an example on the refinement relation in Ex. 2.3.

Example 2.3. *In this example, we illustrate the refinement relation as stated in Def. 2.56. For this, we assume the OCFs κ and κ' as given in Tab. 5 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
6	-
5	pbf
4	$pb\overline{f}$
3	$\overline{p}b\overline{f}$
2	$\overline{p}bf$, $p\overline{b}f$
1	$\overline{p}\overline{b}f$, $p\overline{b}\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$

$\kappa'(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
6	-
5	-
4	-
3	pbf
2	$\overline{p}b\overline{f}$, $pb\overline{f}$
1	$\overline{p}bf$, $p\overline{b}f$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}\overline{b}f$, $p\overline{b}\overline{f}$

Table 5: OCF κ, κ' over signature Σ_{Tweety}, where $\kappa \sqsubseteq \kappa'$.

We see that κ refines κ', i.e. $\kappa \sqsubseteq \kappa'$, since each relation $\omega \preceq_\kappa \omega'$ that holds for κ, also holds for κ'. The differences of $\preceq_\kappa$ and $\preceq_{\kappa'}$ are only with respect to the interpretations $\overline{p}\overline{b}f$, $p\overline{b}\overline{f}$ and $pb\overline{f}$. While for example $\overline{p}\overline{b}f \preceq_{\kappa'} \overline{p}\overline{b}\overline{f}$ and $\overline{p}\overline{b}\overline{f} \preceq_{\kappa'} \overline{p}\overline{b}f$ holds for κ', only $\overline{p}\overline{b}\overline{f} \preceq_\kappa \overline{p}\overline{b}f$ holds for κ. This shows that the relations in κ also hold in κ', but that there can exist relations in κ' that can be omitted in κ due to the refinement.

In summary, we presented ordinal conditional functions as epistemic states and showed how they can be used to represent knowledge by means of a faithful ranking of interpretations and a corresponding belief set. We further stated some of the fundamental properties of OCFs and several equivalence needed in this work. Furthermore, we stated their capability of handling conditional and uncertain knowledge, which can be represented by conditionals – plausible if-then rules. Similar to propositions, we defined the inference of conditionals by OCFs as well as further properties. Lastly, we introduced the concept of minimal models including several properties that state relations of the minimal models of propositions to their conjunctions and disjunctions, respectively. Moreover, we stated the refinement relation of two OCFs κ and κ', that holds if and only if the order of κ is preserved by κ'.

3 Forgetting

Many of the hitherto existing forgetting approaches refer to a specific logic such as propositional logic [Boo54], first order logic [LR94] or answer set programming [Won09, ZF06]. In general, these approaches provide definitions on how to compute the results of forgetting, but less common they state properties capturing the general notions of forgetting. However, there exist attempts on unifying these logic-specific approaches. One of the most prominent is the approach of Delgrande [Del17]. This more general approach is capable of representing several of the logic-specific definitions, which is an important step towards a general framework of forgetting. But even more important is the fact that Delgrande also elaborated properties that reflect the intuitive notions of forgetting according to their opinion. These properties, even if depending on their given definition of forgetting, form a promising basis for elaborating more general properties.

Another approach that attempts to generalize the concept of forgetting is presented by Kern-Isberner et al. in [BKIS+19]. However, both works pursue very different approaches. The first major difference is that Kern-Isberner et al. do provide multiple forgetting definitions. Each of them is motivated by certain cognitive considerations, and therefore describes a different kind of forgetting. Moreover, the there presented kinds of forgetting are not applied to sets of formulas as Delgrande's approach, but more generally to epistemic states, concretely OCFs. Thus, the major goal of [BKIS+19] is to capture the cognitively different kinds of forgetting, instead of unifying the existing logic-specific approaches.

In this section, we will present the general approaches given in [Del17] and [BKIS+19], since they form the basis for our examinations towards a general framework for kinds of forgetting. First, we will present Delgrande's general approach in Section 3.1. Thereby, we cover its basic definition as well as their postulated properties of forgetting and some model theoretical considerations. Finally, we show how their approach can be applied to a specific logic, by expressing Boole's well-known forgetting in propositional logic [Boo54] by means of Delgrande's forgetting. Afterwards, we present in Section 3.2 three of the eight kinds of forgetting given in [BKIS+19], concretely the marginalization, contraction and revision. We chose these three kinds, since we think that they form the most relevant and essential kinds presented by Kern-Isberner et al. The marginalization is the only kind of forgetting that describes forgetting in the sense of forgetting signature elements. Thus, it is especially interesting for a comparison to Delgrande's approach. The contraction in our opinion forms the most intuitive and direct kind of forgetting, since it describes the removal of a certain belief. Furthermore, it is of particular interest since the concept of contraction describes one of the three fundamental belief change operators stated in AGM theory. It also forms the basis for some of the other kinds presented in [BKIS+19]. Lastly, we discuss the concept of revision as a kind of forgetting. What is special about revisions in this context is the fact that they form the only kind of forgetting, whose actual intuition does not describe the forgetting, but the incorporation of information into our present beliefs. In addition to this, they are of particular interest since they also describe one of the three fundamental belief change operators stated in AGM theory.

3.1 Delgrande's General Approach of Forgetting

In the last decades, several definitions of forgetting have been developed. One common feature of all those definitions is that they refer to a specific logic, such as first-order logic [LR94], answer set programming [Won09, ZF06], or propositional logic [Boo54]. Delgrande presents a general approach of forgetting [Del17] with the objective to unify the known specific approaches, and to elaborate general properties of forgetting. This general approach is motivated by the idea that forgetting should be performed at the knowledge level. This means that it argues about the knowledge that follows logically from a given set of formulas, instead of their syntactic structure or the chosen kind of representation. This allows the approach to be applicable to all logics with a well-defined Tarskian consequence relation (Def. 2.12) or consequence operator (Def. 2.16), respectively. Beyond that, no further constraints must be fulfilled. In this section, we will first introduce Delgrande's general forgetting approach. Thereby, we state its main properties **(DFP-1)-(DFP-7)**, which Delgrande refers to as elementary and *right*, in the sense that they display the properties generally associated with forgetting by common sense. After this, we discuss some model theoretical considerations that give further insights on this approach, and furthermore allow us to compare it to other kinds of forgetting in the later sections. Finally, we will present how Delgrande's approach can be applied in propositional logic and how it relates to the already established definition of forgetting in propositional logic as presented by Boole [Boo54].

$\mathcal{F}$ – a general forgetting operator. Motivated by the above-mentioned considerations, Delgrande defines forgetting as a reduction of the logic's language such that the knowledge that should be forgotten can no longer be inferred, since it is not part of the language anymore. The reduction of the language can be achieved by removing certain elements from the underlying signature. Therefore, Delgrande's approach can also be understood as a signature reduction, which means that instead of forgetting a certain formula, we forget the signature elements themselves. This is contrary to most other approaches, which are generally applied to elements of the language. If we for example believe that penguins are able to fly and then come to know that they are actually not, Delgrande's approach would forget about penguins and the ability to fly in general, such that we are not able to argue about them anymore, whereas forgetting just our present beliefs about penguins does not prevent us from doing so. Thus, intuitively Delgrande's approach forgets about the objects and concepts of our world, instead of beliefs about them. Formally, this kind of forgetting is defined in Def. 3.1. In the following, we omit the subscript of the consequence operator Cn_Σ when the signature is clearly given by the context.

Definition 3.1. [Del17] *Let Σ and P be signatures, $\mathcal{L}_{\Sigma\setminus P}$ a language with corresponding consequence operator Cn_Σ, then* forgetting a signature P *in a set of formulas $\Gamma \subseteq \mathcal{L}_\Sigma$ is defined as*

$$\mathcal{F}(\Gamma, P) = Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus P}.$$

When forgetting a signature P in a set of formulas Γ, the deductively closed set of conclusions $Cn_\Sigma(\Gamma)$ with respect to the original signature Σ is determined first, because we want to argue about the knowledge that can be inferred, instead of the formulas themselves. The goal of forgetting the signature elements P from Γ is then to result in a deductively closed set of formulas that does not contain any knowledge mentioning P. Therefore, those conclusions that mention elements of P will be removed by intersecting the deductive closure $Cn_\Sigma(\Gamma)$ with the reduced language $\mathcal{L}_{\Sigma\setminus P}$. The result of forgetting P in Γ is then the knowledge that can be inferred deductively from Γ and does not mention elements of P. Furthermore, we want to point out that forgetting the signature P cannot directly be applied to the set of formulas Γ. Determining the deductive closure before the intersection with the reduced language is crucial. If we assume that $\mathcal{F}(\Gamma, P)$ would reduce Γ to the formulas in $\mathcal{L}_{\Sigma\setminus P}$ first, and then determine the deductive closure of the remaining formulas, there would exist formulas $\varphi \in \mathcal{L}_{\Sigma\setminus P}$ in the reduced language that are a consequence of Γ, but not of $\Gamma\cap\mathcal{L}_{\Sigma\setminus P}$. By changing the order, the forgetting operator would even forget consequences that are part of the reduced language, and would therefore forget more than actually intended by Def. 3.1. In the worst case, Γ only contains formulas mentioning elements in P. Thus, intersecting Γ with the reduced language would result in an empty set of formulas, from which only tautologies could be inferred. This states that Delgrande's definition of forgetting must always be applied to a belief set, and can especially not be applied to a knowledge base.

One of the main goals of the general approach is the unification of the logic-specific forgetting approaches, which requires the comparability of their results in order to verify, if the general approach is able to model the more specific ones. Many forgetting approaches in specific logics, such as Boole's approach in propositional logic [Boo54], consider forgetting in the sense of forgetting formulas, and therefore do not reduce the language to a subsignature. This makes it difficult to compare their results to those of the general approach stated above. To be able to compare the results nevertheless, Delgrande provides a second definition of forgetting (Def. 3.2), expressing the resulting belief set in the original signature Σ. For this the deductive closure with respect to the original signature Σ will be determined for the result of forgetting P in Γ.

Definition 3.2. [Del17] *Let Σ and P be signatures and $\mathcal{L}_\Sigma$ a language with corresponding consequence operator Cn_Σ, then* forgetting a signature P in the original signature Σ *in a set of formulas $\Gamma \subseteq \mathcal{L}_\Sigma$ is defined as*

$$\mathcal{F}_O(\Gamma, P) = Cn_\Sigma(\mathcal{F}(\Gamma, P)).$$

By doing so, the result of forgetting will again contain formulas mentioning elements of the forgotten signature P. At this point the question may arise, whether forgetting as described in Def. 3.2 still captures the idea that it should not be possible to infer any knowledge mentioning elements of the forgotten signature P. However, since we know from Lem. 2.17 that the deductive closure of a set of formulas Γ is equivalent to Γ itself, we can further conclude that both the forgetting with respect to the reduced signature $\Sigma \setminus P$ and to the original signature Σ are equivalent with respect to Σ (Lem. 3.3).

Lemma 3.3. *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P a signature, then the following holds:*

$$\mathcal{F}_O(\Gamma, P) \equiv \mathcal{F}(\Gamma, P)$$

Therefore, we know that $\mathcal{F}_O(\Gamma, P)$ might contain formulas mentioning elements in P, but since the above-stated equivalence holds, these formulas do not provide any new knowledge that could not already be inferred by the result of forgetting P in Γ. Consequently, $\mathcal{F}_O(\Gamma, P)$ can be regarded as forgetting P in Γ as well. Further, we know that if a forgetting approach for a specific logic results in formulas equivalent to $\mathcal{F}_O(\Gamma, P)$, then its result is also equivalent to $\mathcal{F}(\Gamma, P)$. In [Del17], Delgrande shows the connection of their approach to some of the established forgetting operators in specific logics. Since these will not be part of this work, with exception of Boole's forgetting in propositional logic, we will refer to [Del17] for more information.

Besides the unification of the specific approaches, another major goal of Delgrande's approach is the elaboration of general properties of forgetting. Most logic-specific approaches only state ways of computing the result of forgetting, instead of discussing properties that each forgetting approach should satisfy. In Th. 3.4, the properties of this general approach are given. Delgrande emphasizes that these properties are elementary and *right* [Del17], in the sense that they display the properties associated with forgetting by common sense. Thus, we will refer to these properties as Delgrande's forgetting postulates **(DFP-1)**-**(DFP-7)**. Note that the postulates can also be found in Appendix A.1 for faster access.

Theorem 3.4. [Del17] *Let $\mathcal{L}_\Sigma$ be a language over signature Σ and Cn_Σ the corresponding consequence operator, then the following relations hold for all sets of formulas $\Gamma, \Gamma' \subseteq \mathcal{L}$ and signatures P, P'.*

(DFP-1) $\Gamma \models \mathcal{F}(\Gamma, P)$

(DFP-2) *If* $\Gamma \models \Gamma'$, *then* $\mathcal{F}(\Gamma, P) \models \mathcal{F}(\Gamma', P)$

(DFP-3) $\mathcal{F}(\Gamma, P) = Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma, P))$

(DFP-4) *If* $P' \subseteq P$, *then* $\mathcal{F}(\Gamma, P) = \mathcal{F}(\mathcal{F}(\Gamma, P'), P)$

(DFP-5) $\mathcal{F}(\Gamma, P \cup P') = \mathcal{F}(\Gamma, P) \cap \mathcal{F}(\Gamma, P')$

(DFP-6) $\mathcal{F}(\Gamma, P \cup P') = \mathcal{F}(\mathcal{F}(\Gamma, P), P')$

(DFP-7) $\mathcal{F}(\Gamma, P) = \mathcal{F}_O(\Gamma, P) \cap \mathcal{L}_{\Sigma \setminus P}$

Before we continue explaining the above-stated postulates, we want to comment on Delgrande's opinion of **(DFP-1)**-**(DFP-7)** displaying the *right* properties that are associated with forgetting by common sense. In our opinion, **(DFP-1)**-**(DFP-7)** might display the *right* properties when assuming forgetting as a reduction of the language, or as forgetting signature elements, respectively. However, this is clearly not the only kind of forgetting, neither from an intuitive nor from a cognitive perspective. If **(DFP-1)**-**(DFP-7)** also display the *right* properties for other kinds of forgetting is still to be investigated. For this we refer to the later sections.

Even though these properties specifically refer to the definition of Delgrande's general approach (Def. 3.1), they can be used as a basis for discussing and developing more general postulates that do not require a specific definition of forgetting, regardless of how general this definition might be. In the further course, we explain each of the postulates **(DFP-1)**-**(DFP-7)**.

(DFP-1) describes the monotony of forgetting. This means that forgetting a signature P in a set of formulas Γ cannot result in new consequences. Every consequence of $\mathcal{F}(\Gamma, P)$ must be a consequence of Γ as well. This property can be traced back to the fact that $\mathcal{F}(\Gamma, P) = Cn(\Gamma) \cap \mathcal{L}_{\Sigma \setminus P}$ is a subset of the conclusions of Γ.

Given two sets of formulas Γ and Γ', where Γ' can be inferred from Γ, **(DFP-2)** states that if the signature P is forgotten in both sets of formulas, then $\mathcal{F}(\Gamma', P)$ must also be inferred by $\mathcal{F}(\Gamma, P)$. When intersecting the consequences $Cn(\Gamma)$ and $Cn(\Gamma')$ with the reduced language $\mathcal{L}_{\Sigma \setminus P}$, those formulas mentioning elements of P will be removed. Due to

$$\Gamma \models \Gamma' \Leftrightarrow Cn(\Gamma') \subseteq Cn(\Gamma), \qquad \text{(Prop. 2.21)}$$

each formula that will be removed from $Cn(\Gamma')$ will also be removed from $Cn(\Gamma)$. Therefore, we know that after forgetting P in Γ and Γ' the following holds:

$$\begin{aligned}
& Cn(\Gamma') \cap \mathcal{L}_{\Sigma \setminus P} \subseteq Cn(\Gamma) \cap \mathcal{L}_{\Sigma \setminus P} & \\
\Leftrightarrow\ & \mathcal{F}(\Gamma', P) \subseteq \mathcal{F}(\Gamma, P) & \text{(Def. 3.1)} \\
\Leftrightarrow\ & Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma', P)) \subseteq Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma, P)) & \textbf{(DFP-3)} \\
\Leftrightarrow\ & Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma, P)) \models Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma', P)) & \text{(Prop. 2.21)} \\
\Leftrightarrow\ & \mathcal{F}(\Gamma, P) \models \mathcal{F}(\Gamma', P) & \textbf{(DFP-3)}
\end{aligned}$$

Formally, this property is based on the monotony of the Cn operator. An interesting implication of **(DFP-2)** is that forgetting in semantically equivalent sets of formulas again results in semantically equivalent sets (Prop. 3.5).

Proposition 3.5. [Del17] *Let $\Gamma, \Gamma' \in \mathcal{L}_\Sigma$ be sets of formulas and P a signature, then the following holds:*

$$\textit{If } \Gamma \equiv \Gamma', \textit{ then } \mathcal{F}(\Gamma, P) \equiv \mathcal{F}(\Gamma', P)$$

Prop. 3.5 follows straightforwardly from **(DFP-2)** and the definition of semantic equivalence (Def. 2.13).

(DFP-3) shows that the general forgetting operator $\mathcal{F}$ captures the idea that forgetting should be performed on the knowledge level by stating that forgetting is not only performed on, but also results in a belief set. The resulting belief set is deductively closed with respect to the reduced signature $\Sigma \setminus P$. This property follows among others from the reflexivity of the presumed Tarskian consequence relation or consequence operator, respectively.

(DFP-4), **(DFP-5)** and **(DFP-6)** state properties arguing about iterated and simultaneous forgetting. From **(DFP-6)** we know that forgetting a signature P

can also be expressed as forgetting two signatures P_1 and P_2 consecutively with $P = P_1 \cup P_2$. This property is based on simple set theoretical considerations. By definition, forgetting $P = P_1 \cup P_2$ in Γ results in

$$\mathcal{F}(\Gamma, P_1 \cup P_2) = Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)}.$$

Since the reduced language $\mathcal{L}_{\Sigma\setminus(P_1\cup P_2)}$ can also be expressed as $\mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}$, the property stated in **(DFP-6)** can easily be derived:

$$\begin{aligned}
Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} &= Cn_\Sigma(\Gamma) \cap (\mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}) \\
&= (Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus P_1}) \cap \mathcal{L}_{\Sigma\setminus P_2} \\
&= \mathcal{F}(\Gamma, P_1) \cap \mathcal{L}_{\Sigma\setminus P_2} && \text{(Def. 3.1)} \\
&= Cn_{\Sigma\setminus P_1}(\mathcal{F}(\Gamma, P_1)) \cap \mathcal{L}_{\Sigma\setminus P_2} && \textbf{(DFP-3)} \\
&= \mathcal{F}(\mathcal{F}(\Gamma, P_1), P_2) && \text{(Def. 3.1)}
\end{aligned}$$

Therefore, we know that each signature $P = P_1 \cup P_2$ can be forgotten consecutively by forgetting P_1 first, and P_2 afterwards. This property also allows us to express forgetting as a sequence of forgetting operations, where each operation forgets a single signature element (Cor. 3.6).

Corollary 3.6. *Let $P = \{\rho_1, \dots, \rho_n\}$ be a signature with $n \in \mathbb{N}$ and $\Gamma \subseteq \mathcal{L}_\Sigma$ a set of formulas.*

$$\mathcal{F}(\Gamma, P) = \mathcal{F}(\dots \mathcal{F}(\mathcal{F}(\Gamma, \{\rho_1\}), \{\rho_2\}) \dots, \{\rho_n\})$$

Due to the commutativity and associativity of the set union, the order in which the signature elements are forgotten is arbitrary, which means that $\mathcal{F}$ satisfies commutativity and associativity as well (Cor. 3.7).

Corollary 3.7. [Del17] *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P_1, P_2, P_3 signatures, then the following equations hold:*

$$\begin{aligned}
\mathcal{F}(\mathcal{F}(\Gamma, P_1), P_2) &= \mathcal{F}(\mathcal{F}(\Gamma, P_2), P_1) && \text{(Commutativity)} \\
\mathcal{F}(\mathcal{F}(\Gamma, P_1), P_2 \cup P_3) &= \mathcal{F}(\mathcal{F}(\Gamma, P_1 \cup P_2), P_3) && \text{(Associativity)}
\end{aligned}$$

(DFP-5) states that signatures cannot only be forgotten iteratively as stated in **(DFP-6)**, but also simultaneously. Thereby, **(DFP-5)** can be derived similarly to **(DFP-6)**, by means of $\mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} = \mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}$ and set theoretical considerations:

$$\begin{aligned}
\mathcal{F}(\Gamma, P_1 \cup P_2) &= Cn_\Sigma(\Gamma) \cap (\mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}) && \text{(Def. 3.1)} \\
&= (Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus P_1}) \cap (Cn_\Sigma(\Gamma) \cap \mathcal{L}_{\Sigma\setminus P_2}) \\
&= \mathcal{F}(\Gamma, P_1) \cap \mathcal{F}(\Gamma, P_2) && \text{(Def. 3.1)}
\end{aligned}$$

In analogy to Cor. 3.6 we can express the forgetting of P as the intersection of simultaneously performed forgetting operations, where each operation is applied to a single signature element in P (Cor. 3.8).

Corollary 3.8. *Let $P = \{\rho_1, \dots, \rho_n\}$ be a signature with $n \in \mathbb{N}$ and $\Gamma \subseteq \mathcal{L}_\Sigma$ a set of formulas.*

$$\mathcal{F}(\Gamma, P) = \bigcap_{i=1}^{n} \mathcal{F}(\Gamma, \{\rho_i\})$$

(DFP-4) states that forgetting two signatures P' and P consecutively results in the same beliefs as just forgetting P, if $P' \subseteq P$ holds. Thus, previously forgetting a smaller signature, has no effect on the result of forgetting a greater signature afterwards. Moreover, forgetting P after P', only removes the remaining signature elements $P \setminus P'$. This can also be expressed by means of **(DFP-6)**, since $P = P' \cup (P \setminus P')$:

$$\begin{aligned}
\mathcal{F}(\mathcal{F}(\Gamma, P'), P) &= \mathcal{F}(\mathcal{F}(\Gamma, P'), (P' \cup (P \setminus P'))) \\
&= \mathcal{F}(\mathcal{F}(\mathcal{F}(\Gamma, P'), P'), (P \setminus P')) && \textbf{(DFP-6)} \\
&= \mathcal{F}(\mathcal{F}(\Gamma, P'), (P \setminus P')) && (P' \subseteq P', \textbf{(DFP-4)})
\end{aligned}$$

We state this property in Cor. 3.9 below.

Corollary 3.9. *Let P and P' be signatures and $\Gamma \subseteq \mathcal{L}_\Sigma$ a set of formulas.*

$$\textit{If } P' \subseteq P, \textit{ then } \mathcal{F}(\Gamma, P) = \mathcal{F}(\mathcal{F}(\Gamma, P'), P \setminus P')$$

As already seen in the equation stated above, we can conclude from **(DFP-4)** that forgetting the same signature twice is just the same as forgetting it once, which corresponds to the intuition that forgetting something we are not aware of does not affect our present beliefs. This forms a special case of **(DFP-4)** that shows that forgetting is idempotent (Cor. 3.10).

Corollary 3.10. [Del17] *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P a signature. $\mathcal{F}$ satisfies* idempotence.

$$\mathcal{F}(\Gamma, P) = \mathcal{F}(\mathcal{F}(\Gamma, P), P) \qquad \text{(Idempotence)}$$

The last postulate **(DFP-7)** describes that we can obtain the result of forgetting in the reduced signature $\mathcal{F}(\Gamma, P)$, by removing all formulas mentioning elements from P in the result of forgetting in the original signature $\mathcal{F}_O(\Gamma, P)$. Thus, the intersection with $\mathcal{L}_{\Sigma \setminus P}$ acts as the inverse operation to Cn_Σ. Delgrande provides proofs for all properties **(DFP-1)**-**(DFP-7)** in [Del17], but we think that the proof of **(DFP-7)** is not elaborated enough and misses too many steps that are crucial for understanding it. This is why we provide a more detailed and at the same time even shorter proof of **(DFP-7)**.

Proof of **(DFP-7)**. Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P be a signature.

$$\begin{aligned}
\mathcal{F}_O(\Gamma, P) \cap \mathcal{L}_{\Sigma \setminus P} &= Cn_\Sigma(\mathcal{F}(\Gamma, P)) \cap \mathcal{L}_{\Sigma \setminus P} && \text{(Def. 3.2)} \\
&= \mathcal{F}(\mathcal{F}(\Gamma, P), P) && \text{(Def. 3.1)} \\
&= \mathcal{F}(\Gamma, P) && \text{(Cor. 3.10)}
\end{aligned}$$

□

Model theoretical considerations. After providing the general definitions and properties of Delgrande's forgetting approach, we further want to elaborate some of its model theoretical considerations as presented in [Del17]. By this, we will gain a better understanding of this approach, and furthermore be able to compare it to other approaches by means of their models. Before we argue about the models of $\mathcal{F}(\Gamma, P)$ we further have to introduce two definitions, that allow us to argue about interpretations over different signatures Σ', Σ with $\Sigma' \subseteq \Sigma$. First, we define the elementary equivalence of two interpretations $\omega, \omega' \in \Omega_\Sigma$ that states their equivalence with the exception of signature elements P.

Definition 3.11. [Del17] *Let Σ and P be signatures. Two interpretations $\omega, \omega' \in \Omega_\Sigma$ are* elementary equivalent with the exception of the signature elements P, *denoted as*

$$\omega \equiv_P \omega',$$

if and only if they agree on the interpretation of all signature elements in $\Sigma \setminus P$.

Given the definition of elementary equivalence, we can define the reduct and expansion of models in Def. 3.12

Definition 3.12. [Del17] *Let $\Sigma' \subseteq \Sigma$ be signatures and $\varphi \in \mathcal{L}_\Sigma$, $\varphi' \in \mathcal{L}_{\Sigma'}$ formulas with $\varphi \notin \mathcal{L}_{\Sigma'}$. The* reduction to Σ' *of models $[\![\varphi]\!]_\Sigma$ is defined as*

$$([\![\varphi]\!]_\Sigma)_{|\Sigma'} = \{\omega' \in \Omega_{\Sigma'} \mid \textit{there exists } \omega \in [\![\varphi]\!]_\Sigma \textit{ with } \omega \models \omega'\}.$$

The expansion to Σ *of models $[\![\varphi']\!]_{\Sigma'}$ is defined as*

$$([\![\varphi']\!]_{\Sigma'})_{\uparrow\Sigma} = \bigcup_{\omega' \in [\![\varphi']\!]_{\Sigma'}} \omega'_{\uparrow\Sigma},$$

where $\omega'_{\uparrow\Sigma} = \{\omega \in \Omega_\Sigma \mid \omega \models \omega'\}$.

The reduct of the models $[\![\varphi]\!]_\Sigma$ to a subsignature Σ' contains all interpretations $\omega' \in \Omega_{\Sigma'}$ that are satisfied by a model of φ. For each of these interpretations ω' we know that they must agree on the interpretation of all elements in Σ' with a model $\omega \in [\![\varphi]\!]_\Sigma$. Thus, ω' exactly corresponds to ω when omitting the elements $\Sigma \setminus \Sigma'$ (Lem. 3.13).

Lemma 3.13. *Let $\Sigma' \subseteq \Sigma$ be signatures and $\omega \in \Omega_\Sigma$, $\omega' \in \Omega_{\Sigma'}$ interpretations, then the following holds:*

$$\omega \models \omega' \Leftrightarrow \omega_{|\Sigma'} \equiv \omega'$$

This allows us to express multiple iteratively performed reductions to subsignatures Σ' and Σ'' by a single reduct, if $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ holds (Lem. 3.14), which exactly corresponds to the property stated by **(DFP-4)**.

Lemma 3.14. *Let $\varphi \in \mathcal{L}_\Sigma$ be a formula and $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ subsignatures, then the following holds:*

$$([\![\varphi]\!]_{|\Sigma'})_{|\Sigma''} = [\![\varphi]\!]_{|\Sigma''}$$

The expansion of models $[\![\varphi']\!]_{\Sigma'}$ to Σ, where $\Sigma' \subseteq \Sigma$, contains all interpretations $\omega \in \Omega_\Sigma$ that satisfy a model of φ'. This can be described as mapping each model ω' of φ' to the set of interpretations $\omega \in \Omega_\Sigma$ that extend ω' by every possible interpretation of the elements in $\Sigma \setminus \Sigma'$. Therefore, the expansion of the models $[\![\varphi']\!]_{\Sigma'}$ exactly corresponds to the models of φ' with respect to Σ (Lem. 3.15).

Lemma 3.15. *Let $\varphi' \subseteq \mathcal{L}_{\Sigma'}$ be a formula and $\Sigma' \subseteq \Sigma$ a subsignature, then the following holds:*

$$[\![\varphi']\!]_{\uparrow\Sigma} = [\![\varphi']\!]_{\Sigma}$$

Since all interpretations $\omega \in \omega'_{\uparrow\Sigma}$ agree on the interpretation of Σ' but differ on the remaining values $\Sigma \setminus \Sigma'$, we know that by means of a resolution their disjunction must be equivalent to ω' (Lem. 3.16).

Lemma 3.16. *Let $\omega' \in \Omega_{\Sigma'}$ be an interpretation and $\Sigma' \subseteq \Sigma$ a subsignature, then the following holds:*

$$\bigvee_{\omega \in \omega'_{\uparrow\Sigma}} \omega \equiv \omega'$$

We think that understanding the relations of models to their reduct and expansion is crucial for understanding the relations of forgetting in the reduced and in the original signature. Therefore, we want to further illustrate them by giving a concrete example in Ex. 3.1.

Example 3.1. *In this example, we illustrate how the models of a formula relate to their reduct and expansion. For this, we consider the formulas*

$$\begin{aligned} \varphi' &\equiv p \in \mathcal{L}_{\{p\}}, \\ \varphi &\equiv (b \to f) \wedge (p \to b) \in \mathcal{L}_{\Sigma_{Tweety}} \end{aligned}$$

in propositional logic, where $\{p\} \subseteq \Sigma_{Tweety}$. The corresponding models $[\![\varphi']\!]_{\{p\}}$ and $[\![\varphi]\!]_{\Sigma_{Tweety}}$ are given in Tab. 6 below.

$[\![\varphi']\!]_{\{p\}}$	p
ω'_0	*true*

$[\![\varphi]\!]_{\Sigma_{Tweety}}$	p	b	f
ω_0	*false*	*false*	*false*
ω_1	*true*	*false*	*false*
ω_2	*true*	*true*	*false*
ω_3	*true*	*true*	*true*

Table 6: Models of $\varphi' \equiv p$ and $\varphi \equiv (b \to f) \wedge (p \to b)$ with respect to their corresponding signatures $\{p\}$ and Σ_{Tweety}.

When we determine the expansion of $[\![\varphi']\!]_{\{p\}}$ to Σ_{Tweety}, we obtain all interpretations $\omega \in \Omega_{Tweety}$ with $\omega \models \omega'$, for each $\omega' \in [\![\varphi']\!]_{\{p\}}$ (Def. 3.12). Since $[\![\varphi']\!]_{\{p\}}$ only consists of a single model, namely p, we concretely obtain all interpretations $\omega \in \Sigma_{Tweety}$ that assign p to true *as seen in Tab. 7 below.*

$(\llbracket\varphi'\rrbracket_{\{p\}})_{\uparrow\Sigma_{Tweety}}$	p	b	f
$\omega'_{0\uparrow}$	***true***	*false*	*false*
$\omega'_{1\uparrow}$	***true***	*false*	*true*
$\omega'_{2\uparrow}$	***true***	*true*	*false*
$\omega'_{3\uparrow}$	***true***	*true*	*true*

$(\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}}$	p	b
$\omega_{0\vert}$	*false*	*false*
$\omega_{1\vert}$	*true*	*false*
$\omega_{2\vert}$	*true*	*true*

Table 7: Expansion of $\llbracket\varphi'\rrbracket_{\{p\}}$ to Σ_{Tweety} and reduction of $\llbracket\varphi\rrbracket_{\Sigma_{Tweety}}$ to the subsignature $\{p, b\}$.

Therefore, the expansion $(\llbracket\varphi'\rrbracket_{\{p\}})_{\uparrow\Sigma_{Tweety}}$ *equals the models of* p *in* Σ_{Tweety}*, i.e.* $(\llbracket\varphi'\rrbracket_{\{p\}})_{\Sigma_{Tweety}}$*, as stated in Lem. 3.15. Furthermore, we know from Lem. 3.16 that their disjunction is equivalent to* p*:*

$$\bigvee_{\omega\in p_{\uparrow\Sigma_{Tweety}}} \omega = p\overline{b}\overline{f} \vee p\overline{b}f \vee pb\overline{f} \vee pbf \equiv p$$

The reduction of $\llbracket\varphi\rrbracket_{\Sigma_{Tweety}}$ *to the subsignature* $\{p, b\}$ *(Tab. 7) contains all interpretations* $\omega' \in \Omega_{\{p,b\}}$ *that are satisfied by a model* $\omega \in \llbracket\varphi\rrbracket_{\Sigma_{Tweety}}$*. Thus, we obtain all interpretations* ω' *that match a model* ω *when omitting* f*. When we now further reduce* $(\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}}$ *to the subsignature* $\{p\}$*, we obtain the interpretations stated in Tab. 8 below.*

$((\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}})_{\vert\{p\}}$	p
ω_0	*false*
ω_1	*true*

Table 8: Reduction of $(\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}}$ to subsignature $\{p\}$.

The further reduction of $(\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}}$ *to* $\{p\}$ *results in the same interpretations as the reduction of* $\llbracket\varphi\rrbracket_{\Sigma_{Tweety}}$ *to* $\{p\}$*, and therefore we see that*

$$((\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p,b\}})_{\vert\{p\}} = \{p, \overline{p}\} = (\llbracket\varphi\rrbracket_{\Sigma_{Tweety}})_{\vert\{p\}}$$

holds. Thus, first reducing the models to $\{p, b\}$*, and to* $\{p\}$ *afterwards results in the same interpretations as just reducing the models to* $\{p\}$ *as stated in Lem. 3.14.*

By means of the reduct and expansion of models, we can now state in Th. 3.17 how the models after forgetting P in Γ correspond to those of Γ before forgetting.

Theorem 3.17. [Del17] *Let* $\Gamma \subseteq \mathcal{L}_\Sigma$ *be a set of formulas and* $\Sigma' \subseteq \Sigma$ *a subsignature, then the following equations hold:*

1. $\llbracket\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')\rrbracket_{\Sigma'} = (\llbracket\Gamma\rrbracket_\Sigma)_{\vert\Sigma'}$
2. $\llbracket\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')\rrbracket_\Sigma = ((\llbracket\Gamma\rrbracket_\Sigma)_{\vert\Sigma'})_{\uparrow\Sigma}$

We see that the models of $\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')$ with respect to the reduced signature Σ' correspond to the reduct of $[\![\Gamma]\!]$ to Σ', while the models of $\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')$ with respect to Σ correspond to the expansion of the previously reduced models of Γ. This illustrates that forgetting only affects the prior models by removing the forgotten elements, and that in case of forgetting in the original signature, the reduced models are expanded by the forgotten signature elements, such that the expanded models all agree on the interpretation of Σ', but differ on the interpretation of $\Sigma \setminus \Sigma'$. However, according to Lem. 3.16 they are still equivalent to those of the reduced signature. Thus, the models of forgetting with respect to Σ can also be described by means of the elementary equivalence as stated in Th. 3.18.

Theorem 3.18. [Del17] *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P a signature.*

$$[\![\mathcal{F}(\Gamma, P)]\!]_\Sigma = \{\omega \in \Omega_\Sigma \mid \textit{there exists } \omega' \in [\![\Gamma]\!]_\Sigma \textit{ with } \omega \equiv_P \omega'\}$$

Finally, we know that the models of forgetting in the original signature are equal to the expansion of the models of forgetting in the reduced signature (Cor. 3.19).

Corollary 3.19. *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and P a signature, then the following holds:*

$$[\![\mathcal{F}_O(\Gamma, P)]\!]_\Sigma = ([\![\mathcal{F}(\Gamma, P)]\!]_{\Sigma'})_{\uparrow\Sigma}$$

Forgetting in Propositional Logic. As mentioned above, the generalization of the different logic specific forgetting approaches is the main intention behind Delgrande's attempt of defining a general forgetting operator. In [Del17], they presented how their approach can be applied to several logics in order to realize existing forgetting approaches in them. In this last paragraph, we want to exemplary show how Delgrande's forgetting definition can be applied to propositional logic. We do so by comparing it to one of the most well-known forgetting approaches in propositional logic that was presented by Boole in [Boo54]. Boole's definition of forgetting forms the basis for many other forgetting approaches, and therefore is very fundamental. In the following, we will first discuss Boole's definition of forgetting in propositional logic by examining its result in contrast to the original formula and the effect of forgetting on its models. Afterwards, we state the equivalence of both forgetting operators together with the conclusions that can be drawn from it.

Boole describes forgetting in propositional logic in the sense of forgetting an atomic proposition from a formula. At first glance, this might seem very similar to the idea of Delgrande, because atomic propositions can also be regarded as signature elements (Def. 2.2), but both definitions differ in many aspects. Unlike the general forgetting approach, forgetting in propositional logic is not performed on the knowledge level, i.e. it is neither performed on nor does it result in a belief set. Forgetting in propositional logic is purely syntactic, concluding that the forgetting of two equivalent formulas does not necessarily result in the same formula and that forgetting does neither reduce the signature nor the language. Def. 3.20 states the definition of syntactically forgetting an atomic proposition ρ in a formula φ, which is done by substituting ρ by $\bot$ and $\top$ and combining the results disjunctively.

Definition 3.20. [Boo54] *Let $\varphi \in \mathcal{L}_\Sigma$ be a formula and $\rho \in \mathcal{L}_\Sigma$ be an atom.* Forgetting *ρ in φ is then defined as*

$$forget(\varphi, \rho) = \varphi[\rho/\top] \vee \varphi[\rho/\bot],$$

where $\varphi[\rho/\top]$ denotes the substitution of ρ by $\top$, and $\varphi[\rho/\bot]$ the substitution by $\bot$.

The idea behind this procedure is that the resulting formula should be satisfied by the same interpretations as before, but act invariantly towards the interpretation of ρ since it is no longer mentioned in the formula. The substitutions $\varphi[\rho/\top]$ and $\varphi[\rho/\bot]$ describe the situations in which φ is satisfied by an interpretation that interprets ρ as *true* or *false*, respectively. Since the interpretation of ρ should not affect the truth value of $forget(\varphi, \rho)$, it is sufficient that either $\varphi[\rho/\top]$ or $\varphi[\rho/\bot]$ is *true*.

In the following, we want to illustrate how Boole's forgetting in propositional logic works in detail and how the resulting formula $forget(\varphi, \rho)$ relates to the original formula φ. For this, we consider φ to be in conjunctive normal form (Def. 2.6), which allows us to argue about φ more easily. Regarding a fixed atom ρ, the clauses of φ can be divided into three classes. The first class is represented by formula φ^0 and contains all clauses that do not mention ρ. The second class is represented by formula φ^- and contains all clauses in which ρ occurs as a negative literal. The remaining clauses contain ρ as a positive literal and are denoted by formula φ^+. Given these three classes, φ can also be written as $\varphi^0 \wedge \varphi^- \wedge \varphi^+$. When substituting ρ by $\top$ we obtain

$$(\varphi^0 \wedge \varphi^- \wedge \varphi^+)[\rho/\top] \equiv \varphi^0 \wedge \varphi^-[\rho/\top],$$

because $\varphi^+ \equiv \top$ and all clauses in φ^0 do not mention ρ, which is why the substitution has no effect on φ^0. Substituting ρ by $\bot$ works analogously to the substitution by $\top$. Thus, we obtain

$$(\varphi^0 \wedge \varphi^- \wedge \varphi^+)[\rho/\bot] \equiv \varphi^0 \wedge \varphi^+[\rho/\bot].$$

In conclusion, forgetting in propositional logic can also be expressed as

$$\begin{aligned} forget(\varphi, \rho) &= \varphi[\rho/\top] \vee \varphi[\rho/\bot] \\ &\equiv (\varphi^0 \wedge \varphi^-[\rho/\top]) \vee (\varphi^0 \wedge \varphi^+[\rho/\bot]) \\ &\equiv \varphi^0 \wedge (\varphi^-[\rho/\top] \vee \varphi^+[\rho/\bot]). \end{aligned} \tag{3.1}$$

This shows, that all clauses not mentioning ρ must also be *true* after forgetting, in order to fulfil the whole formula. Regarding those clauses that mention ρ, it is no longer necessary that all of them are *true* after forgetting. It is sufficient that either the clauses mentioning ρ as a positive literal or the clauses mentioning ρ as a negative literal are *true*, which is on par with the intuitive explanation given above.

At this point, we want to go more into detail and demonstrate what would happen if all clauses had to be *true* after forgetting, which would be the case if the disjunction of $\varphi^-[\rho/\top]$ and $\varphi^+[\rho/\bot]$ would be a conjunction instead. In view of this assumption, it would be possible to infer knowledge that could not be inferred before, which in fact would conflict with the fundamental idea of forgetting, that knowledge should be removed without obtaining new information. We illustrate this scenario in Ex. 3.2.

Example 3.2. *Let $\Sigma_{Tweety'} = \{b, p, w\}$ be a signature, where b and p have the same extra-logical meanings as in Ex. 2.1 and w can be understood as* the observed animal has wings. *Further, let*

$$\begin{aligned} \varphi &\equiv (p \to w) \wedge (b \to p) \\ &\equiv (\neg p \vee w) \wedge (\neg b \vee p) \\ &\equiv \varphi^- \wedge \varphi^+ \end{aligned}$$

be a formula in $\mathcal{L}_{\Sigma_{Tweety'}}$ and p the atom we want to forget. The result of forgetting p in φ is then given by

$$\begin{aligned} forget(\varphi, p) &= \varphi[p/\top] && \vee && \varphi[p/\bot] \\ &\equiv \varphi^-[p/\top] && \vee && \varphi^+[p/\bot] \\ &= (\neg p \vee w)[p/\top] && \vee && (\neg b \vee p)[p/\bot] \\ &= (\bot \vee w) && \vee && (\neg b \vee \bot) \\ &\equiv w && \vee && \neg b, \end{aligned}$$

which is equivalent to $b \to w$. Since $b \to w$ could also be inferred from φ, forgetting p did not result in new knowledge. On the first glance, it might seem intuitive that the result of $forget(\varphi, p)$ could also be the conjunction of φ^0, $\varphi^-[p/\top]$ and $\varphi^+[p/\bot]$ instead, requiring that all clauses of the CNF must be fulfilled after forgetting. We want to illustrate why this does not capture the idea of forgetting by assuming $forget(\varphi, p)$ to be equivalent to $\varphi^0 \wedge \varphi^-[p/\top] \wedge \varphi^+[p/\bot]$ instead. Forgetting would then result in

$$\varphi^-[p/\top] \wedge \varphi^+[p/\bot] \equiv w \wedge \neg b,$$

which could not be inferred from φ before forgetting ρ. This contradicts the fundamental idea that forgetting should not result in new knowledge.

The insights on how forgetting changes formulas can further be used to examine the influence of forgetting on the formula's models, which will be necessary to show that the results of Boole's forgetting are equivalent to the results of the general approach when applied to propositional logic. In the following, we want to examine the resulting models in detail. Considering the original proposition $\varphi \equiv \varphi^0 \wedge \varphi^- \wedge \varphi^+$ to be in CNF again, the models of φ are then given by

$$[\![\varphi]\!] = [\![\varphi^0]\!] \cap [\![\varphi^-]\!] \cap [\![\varphi^+]\!]$$

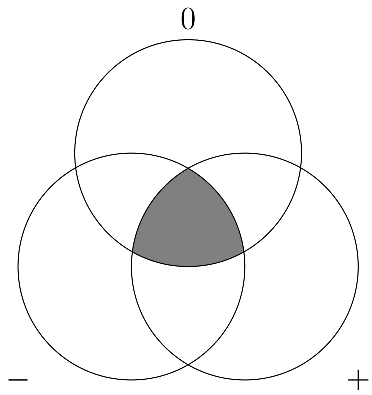
according to Lem. 2.10. In order to determine the models after forgetting, we consider the equivalence $forget(\varphi, \rho) \equiv \varphi^0 \wedge (\varphi^-[\rho/\top] \vee \varphi^+[\rho/\bot])$ (Eq. 3.1), which allows us to argue about the resulting models and compare them to the original models more easily. $\varphi^0 \wedge (\varphi^-[\rho/\top] \vee \varphi^+[\rho/\bot])$ shows that all models have to satisfy φ^0 due to the conjunction, and at least $\varphi^-[\rho/\top]$ or $\varphi^+[\rho/\bot]$ due to the disjunction. Since φ^0 is not affected by the forgetting, the models of φ^0 remain unchanged. Unlike φ^0, the clauses of φ^- and φ^+ are affected by the substitutions, and therefore their models change. When we want to determine the models of φ^- after forgetting, we have to examine how the clauses of φ^- change due to it. For each clause in φ^- we know that it is of the form $(\neg\rho \vee x_0 \vee x_1 \vee \cdots \vee x_n)$, where $(x_0, \ldots, x_n)_{n\in\mathbb{N}_0}$ are arbitrary literals. We assume that each atom only occurs once per clause. When substituting ρ by $\top$, the resulting clauses are of the form $(x_0 \vee x_1 \vee \cdots \vee x_n)$. The only point where the clauses differ is the occurrence of ρ. This means, the original models can be extended by those interpretations that agree on all signature elements with the original models, but possibly differ on the interpretation of ρ, since ρ no longer occurs in the formula. Therefore, the models of the clauses in $\varphi^-[\rho/\top]$ are given by

$$[\![\varphi^-[\rho/\top]]\!] = \{\omega' \in \Omega_\Sigma \mid \text{there exists } \omega \in [\![\varphi^-]\!] \text{ with } \omega \equiv_\rho \omega'\}.$$

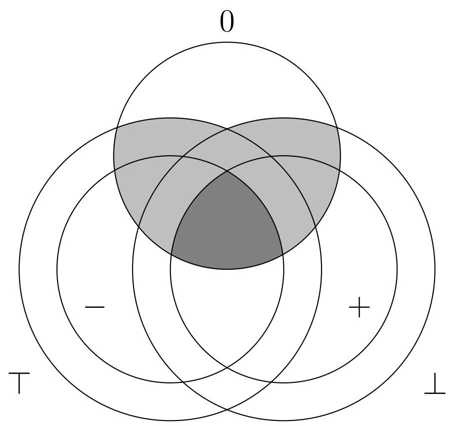
The models of $\varphi^+[\rho/\bot]$ can be obtained analogously. Given the models of φ^0, φ^- and φ^+ after forgetting, Th. 3.21 characterizes the models of $forget(\varphi, \rho)$. The relation between the original models and the models after forgetting is illustrated in Fig. 2.

Theorem 3.21. *Let $\varphi \in \mathcal{L}_\Sigma$ be a proposition in conjunctive normal formal and $\rho \in \mathcal{L}_\Sigma$ an atom, then φ can be expressed as $\varphi \equiv \varphi^0 \wedge \varphi^- \wedge \varphi^+$, where φ^0 contains all clauses not mentioning ρ, φ^- all clauses mentioning ρ as a negative literal, and φ^+ all clauses mentioning ρ as a positive literal. The models of $forget(\varphi, \rho)$ are then given by*

$$[\![forget(\varphi, \rho)]\!] = [\![\varphi^0]\!] \cap ([\![\varphi^-[\rho/\top]]\!] \cup [\![\varphi^+[\rho/\bot]]\!]).$$



(a) $[\![\varphi]\!]$



(b) $[\![forget(\varphi, \rho)]\!]$

Figure 2: Models before and after forgetting an atom ρ in a formula $\varphi \equiv \varphi^0 \wedge \varphi^- \wedge \varphi^+$ in conjunctive normal form illustrated by Venn diagrams. The sets 0, $-$ and $+$ denote the models of φ^0, φ^- and φ^+. The sets $\top$ and $\bot$ denote the models of $\varphi^-[\rho/\top]$ and $\varphi^+[\rho/\bot]$.

After discussing Boole's forgetting in propositional logic in detail, including considerations on the syntactic changes as well as changes of the models, we finally state the equivalence of this approach to the general approach presented by Delgrande (Th. 3.22).

Theorem 3.22. [Del17] *Let $\mathcal{L}_\Sigma$ be the language in propositional logic with signature Σ and let $\rho \in \Sigma$ be an atom.*

$$forget(\varphi, \rho) \equiv \mathcal{F}(\varphi, \rho)$$

Even though the signature is not reduced after applying Boole's forgetting in propositional logic, we say that $forget(\varphi, \rho)$ is equivalent to $\mathcal{F}(\varphi, \rho)$ instead of $\mathcal{F}_O(\varphi, \rho)$, since we know from Lem. 2.17 that $\mathcal{F}(\varphi, \rho)$ and $\mathcal{F}_O(\varphi, \rho)$ are already equivalent with respect to the original signature. For understanding this equivalence more intuitively, we compare the model sets of both forgetting results. Since $\mathcal{F}(\varphi, \rho)$ results in all conclusions that can be inferred from Γ but do not mention ρ, the models that satisfy these conclusions are the models of φ itself together with all interpretations that agree with these models on the interpretation of all signature elements except for ρ. By means of the substitutions in $forget(\varphi, \rho)$, the models of φ are extended by the same interpretations as in $\mathcal{F}(\varphi, \rho)$. This concludes that the models of both forgetting results are equal, and therefore the results must be equivalent. Further, we can obtain the same belief set as obtained by $\mathcal{F}(\varphi, \rho)$ when determining the deductive closure of $forget(\varphi, \rho)$:

$$Cn_{\Sigma \setminus \{\rho\}}(forget(\varphi, \rho)) = \mathcal{F}(\varphi, \rho).$$

In conclusion, we know that Delgrande's general approach is able to model the more specific and very fundamental approach of forgetting in propositional language as presented by Boole [Boo54]. However, this also illustrates that a forgetting definition that results in equivalent beliefs to that of Delgrande does not necessarily have to satisfy the **(DFP-1)**-**(DFP-7)** postulates. In case of $forget(\varphi, \rho)$, we know for instance that the result is not deductively closed. Therefore, **(DFP-3)** and **(DFP-7)** do not hold.

Summary. In this section, we presented Delgrande's general forgetting approach [Del17] that is capable of representing several logic specific forgetting approaches. In contrast to many other approaches, Delgrande describes forgetting in the sense of a signature reduction and states seven properties, which we refer to as Delgrande's forgetting postulates **(DFP-1)**-**(DFP-7)**, and argues that these properties capture the *right* notions of forgetting. By *right* they mean that they display the properties that are usually associated with forgetting by common sense. At this point, we argued that these properties might only be *right*, when assuming forgetting in the sense of forgetting signature elements, but not if we consider other kinds of forgetting. Nonetheless, **(DFP-1)**-**(DFP-7)** form a basis on which more general forgetting postulates can be elaborated. Furthermore, we discussed some of the model theoretical properties and finally showed how the general approach can be applied to propositional logic. Thereby, we showed that it is capable of expressing Boole's well-known forgetting approach in propositional logic [Boo54].

3.2 Kinds of Forgetting in Epistemic States

After presenting Delgrande's general forgetting approach [Del17] in Section 3.1, we will present three of the eight kinds of forgetting presented by Kern-Isberner et al.

[BKIS+19] in the following sections. Other than Delgrande's approach, they specify kinds of forgetting in epistemic states, concretely OCFs, and thereby follow certain cognitive considerations. In the following, we will present the marginalization, contraction and revision as kinds of forgetting in Sections 3.2.1 to 3.2.3.

3.2.1 Marginalization / Focussing

One kind of forgetting that is ever-present in the everyday life of most people is the concept of *focussing*. Even if it might not obviously be related to forgetting at the first sight, focussing on some specific aspects, e.g. while working on a difficult task or studying, always involves the temporary blinding out of irrelevant aspects. The meaning of irrelevance in this context will not be discussed in this work. Instead, we assume the irrelevant aspects to be given and focus on the transformation of the belief state, represented by an OCF, such that the notions of focussing are realized. When we focus on a difficult task like writing a master thesis, we (ideally) do not distract ourselves with thoughts about unrelated things like playing video games or petting dogs. Moreover, since video games and dogs are irrelevant aspects, temporarily forgetting about them does not influence our knowledge relevant for the master thesis. This small example already illustrates the two main notions of focussing:

1. Focussing on relevant aspects changes our beliefs temporarily such that they do not contain any information about irrelevant aspects anymore.

2. Focussing on relevant aspects retains our beliefs about them.

As already described in Section 2.4, OCFs can be understood as a qualitative abstraction of discrete probability distributions, and therefore it is worth mentioning that the notions of focussing are already captured by the concept of marginal distributions. In probability theory, a marginal distribution describes a distribution function that originates from a joint distribution over several variables Σ, but is restricted to a subset $\Sigma' \subseteq \Sigma$ of relevant variables. The impact of the remaining irrelevant variables Σ_{irrel}, where Σ' and Σ_{irrel} form a partitioning of Σ, is then cancelled out in the marginal distribution, since the probability of a certain event over Σ' will be determined by taking all those assignments of Σ into account that agree on the assignment of Σ', but differ on the assignment of Σ_{irrel}. By accumulating those probabilities, we obtain the total probability that a certain event over Σ' occurs. The resulting marginal distribution $P_{|\Sigma'}$ then assigns the same probabilities to all events over Σ' as the prior joint distribution P, but no longer assigns probabilities to events over Σ_{irrel}. Thus, marginal distributions capture the notions of focussing.

Shenoy originally translated the concept of marginal distributions to OCFs in [She91], which in turn builds upon Spohn's work on OCFs [Spo88], and thus captured the notions of focussing qualitatively. However, we will refer to the definition of marginalization as given by Kern-Isberner et al. [BKIS+19], since the examination of the kinds of forgetting presented there are one of the main goals of this thesis. Moreover the notation used by Kern-Isberner et al. suits better for the comparison to Delgrande's forgetting approach [Del17]. The reduction of the regarded random variables to a relevant subset corresponds to the reduction of the signature, meaning that the marginalized OCF will be defined over a subset of relevant signature elements Σ', and thus fulfils the first notion of focussing. The ranks of the marginalized interpretations are similarly determined to the probabilities in the marginal distribution by taking all interpretations over Σ into account that agree on the interpretation of Σ', but differ on the interpretation of the remaining elements $\Sigma \setminus \Sigma'$. Considering those interpretations, the marginalized interpretation is assigned to the smallest rank among them. This way a marginalized interpretation is as plausible as the most plausible interpretation over Σ that satisfies it. The marginalization of an OCF is formally given in Def. 3.23. The rank of formulas after the marginalization follows directly from the marginalization itself and the rank of formulas as given in Def. 2.31.

Definition 3.23 ([BKIS+19])**.** *Let κ be an OCF over signature Σ and $\omega \in \Omega_{\Sigma'}$ an interpretation with $\Sigma' \subseteq \Sigma$. $\kappa_{|\Sigma'}$ is called a* marginalization *of κ to Σ' with*

$$\kappa_{|\Sigma'}(\omega') = \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \omega'\}.$$

Furthermore, this concludes that the most plausible interpretations after the marginalization can be determined by means of reducing the signature of the prior most plausible interpretations from Σ to Σ' (Prop. 3.24).

Proposition 3.24. *Let κ be an OCF over signature Σ and $\Sigma' \subseteq \Sigma$ a subsignature.*

$$\llbracket \kappa_{|\Sigma'} \rrbracket = \llbracket \kappa \rrbracket_{|\Sigma'}$$

Proof of Prop. 3.24.

$$\begin{aligned}
\llbracket \kappa_{|\Sigma'} \rrbracket &= \{\omega' \in \Omega_{\Sigma'} \mid \kappa_{|\Sigma'}(\omega') = 0\} && \text{(Def. 2.30)}\\
&= \{\omega' \in \Omega_{\Sigma'} \mid \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \omega'\} = 0\} && \text{(Def. 3.23)}\\
&= \{\omega' \in \Omega_{\Sigma'} \mid \text{there exist } \omega \in \Omega_\Sigma \text{ with } \omega \models \omega' \text{ and } \kappa(\omega) = 0\} &&\\
&= \{\omega' \in \Omega_{\Sigma'} \mid \text{there exist } \omega \in \Omega_\Sigma \text{ with } \omega \models \omega' \text{ and } \omega \in \llbracket \kappa \rrbracket\} && \text{(Prop. 2.37)}\\
&= \{\omega' \in \Omega_{\Sigma'} \mid \text{there exist } \omega \in \llbracket \kappa \rrbracket \text{ with } \omega \models \omega'\} &&\\
&= \llbracket \kappa \rrbracket_{|\Sigma'} && \text{(Def. 3.12)}
\end{aligned}$$

□

Before we show that also the second notion of focussing is captured by the marginalization, we want to discuss a slight difference between marginalized OCFs and marginal distributions in the way the resulting ranks and probabilities are determined. While summing up the probabilities for all events that agree on a certain variable assignment over Σ' in marginal distributions, we only select the most plausible interpretation that agrees on the interpretation of Σ' in marginalized OCFs. Due to the qualitative abstraction, the marginalization is not able to capture the phenomena that multiple small probabilities in the joint distribution can result in the highest probability in the marginal distribution. From a cognitive perspective this can also be viewed as a rather positive aspect, since it might not always be a pleasant behaviour when multiple small probabilities accumulated form an event more probable than an event that is much likelier to occur than each of the other, especially when the difference between the probabilities magnitudes is rather great. Given the joint distribution over $\Sigma = \{X, Y\}$ in Tab. 9, where both random variables are binary, we see that $(X = 0, Y = 0)$ is the likeliest event, followed by $(X = 1, Y = 0)$, $(X = 1, Y = 1)$ and $(X = 0, Y = 1)$. Note that we denote $X = 0$ by $\overline{x}$, and $X = 1$ by x for a more uniform notation.

$P(X, Y)$	$\omega \in \Omega_\Sigma$
0.10	$\overline{x}y$
0.25	xy
0.30	$x\overline{y}$
0.35	$\overline{x}\overline{y}$

Table 9: Joint discrete distribution over $\Sigma = \{X, Y\}$. The first column $P(X, Y)$ describes the probability assigned to the possible variable assignments of X and Y in the second column.

Ordering the events by their probabilities, we can also represent the joint distribution by an OCF as seen in Tab. 10. Thereby, the variables X and Y the joint distribution is defined over correspond to the signature elements x and y. The possible variable assignments of X and Y correspond to the interpretations Ω_Σ. Note that since the OCF representing the joint distribution is not unique, other OCFs might be possible as well. However, they would all agree on the order of the interpretations.

$\kappa(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
4	-
3	$\overline{x}y$
2	xy
1	$x\overline{y}$
0	$\overline{x}\overline{y}$

Table 10: OCF over $\Sigma = \{x, y\}$ abstracting from the joint distribution in Tab. 9.

Marginalizing both the joint distribution and the OCF to $\Sigma' = \{X\}$ results in the marginal distribution $P_{|\Sigma'}$ and the marginalized OCF $\kappa_{|\Sigma'}$ as seen in Tab. 11 and 12 below.

$P_{\|\Sigma'}(X)$	$\omega \in \Omega_{\Sigma'}$
0.45	$\overline{x}$
0.55	x

Table 11: Marginal discrete distribution $P_{|\Sigma'}$ obtained by marginalizing the joint probability from Tab. 9

$\kappa_{\|\Sigma'}(\omega)$	$\omega \in \Omega_{\Sigma'}$
∞	-
$\vdots$	-
2	-
1	x
0	$\overline{x}$

Table 12: Marginalization of the OCF from Tab. 10.

In the marginal distribution, the summation results in x being likelier than $\overline{x}$, even though the highest probability in the joint distribution is $P(\overline{x}\overline{y}) = 0.35$. The same effect cannot be observed in the marginalized OCF. The selection of the minimal rank results in $\overline{x}$ still being more plausible than x. The order of x and $\overline{x}$ is not inverted, because by abstracting the joint distribution we lose information on the exact values, and therefore we do not know if the interpretations $x\overline{y}$ and xy with ranks 1 and 2 together are more plausible than $\overline{x}\overline{y}$ with rank 0. Despite those differences, Prop. 3.25 shows that the marginalization captures the second notion of focussing nevertheless, since all conditional over Σ' that could be inferred by κ can also be inferred after the marginalization. Since Kern-Isberner et al. did not include the proof of Prop. 3.25 in [BKIS$^+$19], we prove it in the following.

Proposition 3.25. [BKIS$^+$19] *Let κ be an OCF over Σ and $\Sigma' \subseteq \Sigma$, then for each conditional $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$ the following holds:*

$$\kappa_{|\Sigma'} \models (\psi|\varphi) \Leftrightarrow \kappa \models (\psi|\varphi)$$

Proof of Prop. 3.25.

$$\begin{aligned}
& \kappa \models (\psi|\varphi) \\
\Leftrightarrow\ & \kappa(\varphi \wedge \psi) < \kappa(\varphi \wedge \neg\psi) && \text{(Prop. 2.49)} \\
\Leftrightarrow\ & \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma \text{ and } \omega \models \varphi \wedge \psi\} \\
& < \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma \text{ and } \omega \models \varphi \wedge \neg\psi\} && \text{(Def. 2.31)} \\
\Leftrightarrow\ & \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma, \omega' \in \Omega_{\Sigma'} \text{ with } \omega \models \omega' \text{ and } \omega' \models \varphi \wedge \psi\} \\
& < \min\{\kappa(\omega) \mid \omega \in \Omega_\Sigma, \omega' \in \Omega_{\Sigma'} \text{ with } \omega \models \omega' \text{ and } \omega' \models \varphi \wedge \neg\psi\} && \text{(Note)} \\
\Leftrightarrow\ & \min\{\kappa_{|\Sigma'}(\omega) \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega' \models \varphi \wedge \psi\} \\
& < \min\{\kappa_{|\Sigma'}(\omega) \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega' \models \varphi \wedge \neg\psi\} && \text{(Def. 3.23)}
\end{aligned}$$

$$\Leftrightarrow \kappa_{|\Sigma'}(\varphi \wedge \psi) < \kappa_{|\Sigma'}(\varphi \wedge \neg\psi) \qquad \text{(Def. 2.31)}$$
$$\Leftrightarrow \kappa_{|\Sigma'} \models (\psi|\varphi) \qquad \text{(Prop. 2.49)}$$

Note: Since $(\psi|\varphi)$ is already defined over the reduced signature Σ', we know that the predicates $\omega \models \varphi \wedge \psi$ and $\omega \models \omega', \omega' \models \varphi \wedge \psi$ are equivalent. This holds analogously for $\varphi \wedge \neg\psi$.

□

If we consider $\varphi \equiv \top$ in Prop. 3.25, we can explicitly state the preservation of the prior beliefs that only mention elements of the reduced signature Σ' (Lem. 3.26).

Lemma 3.26. *Let κ be an OCF over Σ and $\Sigma' \subseteq \Sigma$, then for each $\varphi \in \mathcal{L}_{\Sigma'}$ the following holds:*

$$\kappa_{|\Sigma'} \models \varphi \Leftrightarrow \kappa \models \varphi$$

Moreover, from Lem. 3.26 we know that if a set of interpretations $\Theta \in \Omega_\Sigma$ satisfies a formula $\varphi \in \mathcal{L}_{|\Sigma'}$, where $\Sigma' \subseteq \Sigma$, then $\Theta_{|\Sigma'}$ satisfies φ as well, since the truth values assigned to $\Sigma \setminus \Sigma'$ do not influence whether φ is true or not (Lem. 3.27).

Lemma 3.27. *Let $\Theta \subseteq \Omega_\Sigma$ and $\Sigma' \subseteq \Sigma$, then for each $\varphi \in \mathcal{L}_{\Sigma'}$ the following holds:*

$$\textit{If } \Theta \models \varphi, \textit{ then } \Theta_{|\Sigma'} \models \varphi$$

After marginalizing an OCF κ to a reduced signature Σ', the resulting OCF $\kappa_{|\Sigma'}$ can be lifted to the original signature Σ again. This is of particular interest when we want to compare the result of forgetting to other belief states or sets that argue about the original signature. Since forgetting in the sense of marginalization is defined as a signature reduction, it is important that lifting to the original signature does not violate the considered notions of focussing described above. This means, that after lifting the marginalized OCF, we still want to be able to infer the exact same beliefs over the reduced signature Σ' as before, and additionally all inferences mentioning elements of $\Sigma \setminus \Sigma'$ should either be tautologies or follow trivially from our beliefs over Σ'. However, we want to mention that even if we use the concept of lifting in the sense of re-introducing certain signature elements to the marginalized OCF, the concepts of lifting and marginalization do not necessarily relate to each other, but can also be regarded independently. In general, lifting describes the process of becoming aware of new objects or concepts of the world, which we usually refer to as signature elements. Thus, these new signature elements will be added to the hitherto signature. Ideally, we want to be unbiased towards the just added signature elements, and therefore neither of their interpretations should be regarded as more plausible than the other, while the relations of the interpretations over the prior signature should be retained. In general, there exist two different notions of lifting. The first notion describes lifting such that each possible OCF κ over signature Σ is considered a lifting of another OCF κ' over signature $\Sigma' \subseteq \Sigma$, if κ' can be obtained by marginalizing κ, i.e. $\kappa_{|\Sigma'} = \kappa'$. By means of this notion, the lifting of an OCF κ' is ambiguous, since there might exist multiple OCFs κ, from which κ' can be originated. However, when we argue about lifting in this work, we always refer to the unique lifting of an OCF κ' as defined in Def. 3.28.

Definition 3.28. *Let κ' be an OCF over signature $\Sigma' \subseteq \Sigma$. A* lifting *of κ' to Σ, denoted by $\kappa'_{\uparrow\Sigma}$, is uniquely defined by $\kappa'_{\uparrow\Sigma}(\omega) = \kappa'(\omega_{|\Sigma'})$ for all $\omega \in \Omega_\Sigma$.*

Here, the lifted OCF $\kappa'_{\uparrow\Sigma}$ is obtained by assigning all interpretations $\omega, \omega' \in \Omega_\Sigma$ with $\omega \equiv_{\Sigma\setminus\Sigma'} \omega'$ to the same rank, namely $\kappa_{|\Sigma'}(\omega_{|\Sigma'})$. This definition exactly corresponds to the lifting as described by Kern-Isberner et al. in [BKIS+19]. Due to the expansion of Σ' to Σ, each interpretation $\omega' \in \Omega_{\Sigma'}$ is mapped to a set of interpretations $\omega \in \Omega$ with $\omega_{|\Sigma'} \equiv \omega'$. This set contains all interpretations that agree on the interpretation of the prior signature elements Σ', but differ on the interpretation of $\Sigma \setminus \Sigma'$. According to Def. 3.28, the rank of each $\omega \in \Omega_\Sigma$ corresponds to the rank of the interpretations $\omega' \in \Omega_{\Sigma'}$ with $\omega_{|\Sigma'} \equiv \omega'$. Therefore, all interpretations that agree on the interpretation of Σ' are assigned to the same rank, realising an unbiased behaviour towards the newly added signature elements. Moreover, the order and ranks of the prior interpretations are preserved.

In the following, we focus on lifting a previously marginalized OCF back to its original signature in order to illustrate the relations between the two concepts. For this, we first show that lifting an OCF κ' over signature Σ' will retain the inferences of all conditionals that are defined over Σ' (Prop. 3.29).

Proposition 3.29. *Let κ' be an OCF over signature $\Sigma' \subseteq \Sigma$, then*

$$\kappa'_{\uparrow\Sigma} \models (\psi|\varphi) \Leftrightarrow \kappa' \models (\psi|\varphi)$$

holds for each conditional $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$.

Proof of Prop. 3.29.

$$\begin{aligned}
&\kappa'_{\uparrow\Sigma} \models (\psi|\varphi) \\
\Leftrightarrow\ &\kappa'_{\uparrow\Sigma}(\varphi\wedge\psi) < \kappa'_{\uparrow\Sigma}(\varphi\wedge\neg\psi) && \text{(Prop. 2.49)}\\
\Leftrightarrow\ &\min\{\kappa'_{\uparrow\Sigma}(\omega) \mid \omega \models \varphi\wedge\psi\} < \min\{\kappa'_{\uparrow\Sigma}(\omega) \mid \omega \models \varphi\wedge\neg\psi\} && \text{(Def. 2.31)}\\
\Leftrightarrow\ &\min\{\kappa'(\omega_{|\Sigma'}) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \varphi\wedge\psi\} \\
&< \min\{\kappa'(\omega_{|\Sigma'}) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \varphi\wedge\neg\psi\} && \text{(Def. 3.28)}\\
\Leftrightarrow\ &\min\{\kappa'(\omega') \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega' \models \varphi\wedge\psi\} \\
&< \min\{\kappa'(\omega') \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega' \models \varphi\wedge\neg\psi\} && \text{(Note)}\\
\Leftrightarrow\ &\kappa'(\varphi\wedge\psi) < \kappa'(\varphi\wedge\neg\psi) && \text{(Def. 3.23)}\\
\Leftrightarrow\ &\kappa' \models (\psi|\varphi) && \text{(Prop. 2.49)}
\end{aligned}$$

Note: Since $\varphi\wedge\psi \in \mathcal{L}_{\Sigma'}$, we know that $\omega_{|\Sigma'} \models \varphi\wedge\psi$ if and only if $\omega \models \varphi\wedge\psi$. We further refer to $\omega_{|\Sigma'}$ as ω'.

□

Since Prop. 3.29 states that lifting an OCF from Σ' to Σ always retains the conditional inferences of the prior OCF, we know that this must especially hold for any previously marginalized OCF $\kappa_{|\Sigma'}$. Therefore, we can also conclude that the first marginalized and then lifted OCF $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ and the initial OCF κ over signature

Σ infer the exact same conditionals over the reduced signature Σ' (Cor. 3.30). This directly concludes from Prop. 3.25 and Prop. 3.29. Thus, the second notion of focussing is retained, even if we lift the marginalized OCF back to the original signature Σ.

Corollary 3.30. *Let κ be an OCF over signature Σ and $\kappa_{|\Sigma'}$ the marginalization of κ to the subsignature $\Sigma' \subseteq \Sigma$, then*

$$\kappa \models (\psi|\varphi) \Leftrightarrow (\kappa_{|\Sigma'})_{\uparrow\Sigma} \models (\psi|\varphi)$$

holds for each $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$

Since the lifted OCF $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ is also capable of arguing about those signature elements that were forgotten by the marginalization, we have to examine which conditionals mentioning elements of $\Sigma \setminus \Sigma'$ are satisfied by it. First, we examine the belief set $Bel((\kappa_{|\Sigma'})_{\uparrow\Sigma})$, e.g. the behaviour of $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ towards formulas over Σ or conditionals of the form $(\psi|\varphi)$ with $\varphi \equiv \top$ respectively. For this, we state in Prop. 3.31, that the most plausible interpretations after lifting an OCF can analogously be determined to those of the marginalization (Prop. 3.24) by extending the signature of the prior most plausible interpretations from Σ' to Σ. Note that the $[\![\cdot]\!]_{\uparrow\Sigma}$ states the extension of the models to Σ as defined in Def. 3.12.

Proposition 3.31. *Let κ' be an OCF over signature $\Sigma' \subseteq \Sigma$.*

$$[\![\kappa'_{\uparrow\Sigma}]\!] = [\![\kappa']\!]_{\uparrow\Sigma}$$

Proof of Prop. 3.31.

$$\begin{aligned}
[\![\kappa']\!]_{\uparrow\Sigma} &= \bigcup_{\omega' \in [\![\kappa']\!]} \{\omega \in \Omega_\Sigma \mid \omega \models \omega'\} && \text{(Def. 3.12)}\\
&= \{\omega \in \Omega_\Sigma \mid \text{there exist } \omega' \in [\![\kappa']\!]_{\Sigma'} \text{ with } \omega \models \omega'\}\\
&= \{\omega \in \Omega_\Sigma \mid \text{there exist } \omega' \in [\![\kappa']\!]_{\Sigma'} \text{ with } \omega_{|\Sigma'} \equiv \omega'\} && \text{(Lem. 3.13)}\\
&= \{\omega \in \Omega_\Sigma \mid \omega_{|\Sigma'} \in [\![\kappa']\!]_{\Sigma'}\} && \text{(Note)}\\
&= \{\omega \in \Omega_\Sigma \mid \kappa'(\omega_{|\Sigma'}) = 0\} && \text{(Prop. 2.37)}\\
&= \{\omega \in \Omega_\Sigma \mid \kappa'_{\uparrow\Sigma}(\omega) = 0\} && \text{(Def. 3.28)}\\
&= [\![\kappa'_{\uparrow\Sigma}]\!] && \text{(Def. 2.30)}
\end{aligned}$$

Note: We know that if there exists an interpretation $\omega' \in [\![\kappa']\!]_{\Sigma'}$ that is equivalent to $\omega_{|\Sigma'}$, then $\omega_{|\Sigma'}$ is included in $[\![\kappa']\!]_{\Sigma'}$ as well, and vice-versa.

□

Due to Prop. 3.31 and the relation between the theory operator Th and the consequence operator Cn (Lem. 2.23), we can express the belief set of $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ by means of the prior beliefs $Bel(\kappa_{|\Sigma'})$ (Prop. 3.32).

Proposition 3.32. *Let κ' be an OCF over signature $\Sigma' \subseteq \Sigma$ and $\kappa'_{\uparrow\Sigma}$ be a lifting of κ' to Σ, then the beliefs of $\kappa'_{\uparrow\Sigma}$ are given by*

$$Bel(\kappa'_{\uparrow\Sigma}) = Cn_{\Sigma}(Bel(\kappa')).$$

Proof of Prop. 3.32.

$$\begin{aligned}
Bel(\kappa'_{\uparrow\Sigma}) &\equiv Th(\llbracket\kappa'_{\uparrow\Sigma}\rrbracket) && \text{(Lem. 2.39)}\\
&\equiv Cn_{\Sigma}\Big(\bigvee_{\omega\in\llbracket\kappa'_{\uparrow\Sigma}\rrbracket} \omega\Big) && \text{(Lem. 2.23)}\\
&\equiv Cn_{\Sigma}\Big(\bigvee_{\omega\in\llbracket\kappa'\rrbracket_{\uparrow\Sigma}} \omega\Big) && \text{(Prop. 3.31)}\\
&\equiv Cn_{\Sigma}\Big(\bigvee_{\omega\in \bigcup_{\omega'\in\llbracket\kappa'\rrbracket} \omega'_{\uparrow\Sigma}} \omega\Big) && \text{(Def. 3.12)}\\
&\equiv Cn_{\Sigma}\Big(\bigvee_{\omega'\in\llbracket\kappa'\rrbracket} \Big(\bigvee_{\omega\in\omega'_{\uparrow\Sigma}} \omega\Big)\Big) && \\
&\equiv Cn_{\Sigma}\Big(\bigvee_{\omega'\in\llbracket\kappa'\rrbracket} \omega'\Big) && \text{(Lem. 3.16)}\\
&\equiv Cn_{\Sigma}\Big(Cn_{\Sigma'}\Big(\bigvee_{\omega'\in\llbracket\kappa'\rrbracket} \omega'\Big)\Big) && \text{(Lem. 2.17)}\\
&\equiv Cn_{\Sigma}(Th(\llbracket\kappa'\rrbracket)) && \text{(Lem. 2.23)}\\
&\equiv Cn_{\Sigma}(Bel(\kappa')) && \text{(Lem. 2.39)}
\end{aligned}$$

□

We further illustrate the relation between marginalization and lifting, in which lifting a previously marginalized OCF back to its original signature does not yield new beliefs over the reduced signature Σ', and only adds beliefs over the re-introduced signature elements that are either tautologies or can already be inferred by the beliefs of the marginalized OCF. If we consider the marginalized OCF $\kappa_{|\Sigma'}$ in Tab. 12 and lift it back to the original signature $\Sigma = \{x, y\}$, we obtain the OCF $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ as given by Tab. 13.

$(\kappa_{\vert\Sigma'})_{\uparrow\Sigma}(\omega)$	$\omega \in \Omega_{\Sigma}$
∞	-
$\vdots$	-
2	-
1	xy, $x\overline{y}$
0	$\overline{x}y$, $\overline{x}\overline{y}$

Table 13: Lifting of the marginalized OCF from Tab. 12 to the original signature $\Sigma = \{x, y\}$.

Given the above-mentioned considerations from Prop. 3.32, the belief set of $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ in Tab. 13 is given by $Cn_{\Sigma}(\overline{x}y \vee \overline{x}\overline{y})$, which can be simplified to $Cn_{\Sigma}(\overline{x})$ by resolution and equals the deductive closure of all interpretations with rank 0 in the marginalized OCF $\kappa_{|\Sigma'}$ (Tab. 12) with respect to the original signature Σ. Note that at this point we make use of the fact that interpretations can also be considered as a conjunction of literals, such that the formula's only model is the interpretation it represents. Further we know $Cn_{\Sigma}(\overline{x}) = Cn_{\Sigma}(Cn_{\Sigma'}(\overline{x}))$, because the deductive closure over Σ also contains all formulas that can be inferred over the subsignature Σ'. Finally, we know that $Cn_{\Sigma}(Cn_{\Sigma'}(\overline{x})) = Cn_{\Sigma}(Bel(\kappa_{|\Sigma'}))$ due to the properties of Bel. Note that even though a belief set is deductively closed by definition, $Cn_{\Sigma}(Bel(\kappa_{|\Sigma'}))$ still adds new formulas to $Bel(\kappa_{|\Sigma'})$, since the belief set is defined over the reduced signature Σ', while Cn_{Σ} determines the deductive closure with respect to the original signature Σ.

This relation shows that $Bel((\kappa_{|\Sigma'})_{\uparrow\Sigma})$ extends $Bel(\kappa_{|\Sigma'})$ by those formulas mentioning elements of $\Sigma \setminus \Sigma'$ that are either tautologies or can be inferred trivially from some formula in $Bel(\kappa_{|\Sigma'})$, e.g. $\overline{x} \models \overline{x} \vee y$ [5]. Resulting in a belief set that cannot make non trivial statements on elements of $\Sigma \setminus \Sigma'$, we consider the second notion of focussing to be fulfilled, and therefore lifting the marginalized OCF can still be regarded as focussing on Σ'. However, keep in mind that focussing and marginalization are still different operations and follow different cognitive considerations as already described above.

Next, we examine the influence of lifting on the behaviour of $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ towards general conditionals $(\psi|\varphi) \in (\mathcal{L}_{\Sigma}|\mathcal{L}_{\Sigma})$ instead of only those where $\varphi \equiv \top$. As already stated in Prop. 3.29, all conditionals $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$ that can be inferred by $\kappa_{|\Sigma'}$ can also be inferred by $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$, but we additionally want $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ to act invariantly towards elements of $\Sigma \setminus \Sigma'$ in order to still capture the notions of focussing. More precisely, for each conditional $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$ with $(\kappa_{|\Sigma'})_{\uparrow\Sigma} \models (\psi|\varphi)$ we want the lifted OCF to infer each conditional $(\psi|\varphi \wedge \xi)$ for $\xi \in \mathcal{L}_{\Sigma\setminus\Sigma'}$ as well. Kern-Isberner et al. touch on this property briefly in [BKIS+19] without going much into detail. This being a fundamental property of lifting a marginalized OCF, we generally formalize it in Prop. 3.33 and prove it afterwards.

Proposition 3.33. *Let κ' be an OCF over $\Sigma' \subseteq \Sigma$, $(\psi|\varphi) \in (\mathcal{L}_{\Sigma'}|\mathcal{L}_{\Sigma'})$ a conditional and $\xi \in \mathcal{L}_{\Sigma\setminus\Sigma'}$ a formula.*

$$\kappa'_{\uparrow\Sigma} \models (\psi|\varphi) \Leftrightarrow \kappa'_{\uparrow\Sigma} \models (\psi|\varphi \wedge \xi)$$

Proof of Prop. 3.33.

$$\begin{aligned}
&\kappa'_{\uparrow\Sigma} \models (\psi|\varphi \wedge \xi) \\
\Leftrightarrow\ &\kappa'_{\uparrow\Sigma}(\psi \wedge \varphi \wedge \xi) < \kappa'_{\uparrow\Sigma}(\neg\psi \wedge \varphi \wedge \xi) && \text{(Prop. 2.49)} \\
\Leftrightarrow\ &\min\{\kappa'_{\uparrow\Sigma}(\omega) \mid \omega \in \Omega_{\Sigma} \text{ with } \omega \models \psi \wedge \varphi \wedge \xi\} \\
&\quad < \min\{\kappa'_{\uparrow\Sigma}(\omega) \mid \omega \in \Omega_{\Sigma} \text{ with } \omega \models \neg\psi \wedge \varphi \wedge \xi\} && \text{(Def. 2.31)}
\end{aligned}$$

[5] We will take up and prove the relation between the two belief sets in the comparison of Delgrande's forgetting approach and the marginalization in Section 4.1.

$$\begin{aligned}
&\Leftrightarrow \min\{\kappa'(\omega_{|\Sigma'}) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \psi \wedge \varphi \wedge \xi\} \\
&\quad < \min\{\kappa'(\omega_{|\Sigma'}) \mid \omega \in \Omega_\Sigma \text{ with } \omega \models \neg\psi \wedge \varphi \wedge \xi\} && \text{(Def. 3.28)} \\
&\Leftrightarrow \min\{\kappa'(\omega') \mid \omega \in \Omega_\Sigma, \omega' \in \Omega_{\Sigma'} \\
&\qquad\quad \text{with } \omega \models \psi \wedge \varphi \wedge \xi \text{ and } \omega' \equiv \omega_{|\Sigma'}\} \\
&\quad < \min\{\kappa'(\omega') \mid \omega \in \Omega_\Sigma, \omega' \in \Omega_{\Sigma'} \\
&\qquad\qquad \text{with } \omega \models \neg\psi \wedge \varphi \wedge \xi \text{ and } \omega' \equiv \omega_{|\Sigma'}\} \\
&\Leftrightarrow \min\{\kappa'(\omega') \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega' \models \psi \wedge \varphi\} \\
&\quad < \min\{\kappa'(\omega') \mid \omega' \in \Omega_{\Sigma'} \text{ with } \omega \models \neg\psi \wedge \varphi\} && \text{(Note)} \\
&\Leftrightarrow \kappa' \models (\psi|\varphi) && \text{(Prop. 2.49)} \\
&\Leftrightarrow \kappa'_{\uparrow\Sigma} \models (\psi|\varphi) && \text{(Prop. 3.29)}
\end{aligned}$$

Note: The minimum is determined over the interpretations over the reduced signature Σ' that satisfy $\omega' \equiv \omega_{|\Sigma'}$, where $\omega \models \psi \wedge \varphi \wedge \xi$. Since $\xi \in \mathcal{L}_{\Sigma\setminus\Sigma'}$, we know that the interpretations considered for the minimum do not change, if we assume $\omega \models \psi \wedge \varphi$ instead. Since $\psi \wedge \varphi \in \mathcal{L}_{\Sigma'}$, we especially know $\omega_{|\Sigma'} \models \psi \wedge \varphi$. Therefore, the conditions $\omega \models \psi \wedge \varphi \wedge \xi$ and $\omega' \equiv \omega_{|\Sigma'}$ can be simplified to $\omega' \models \psi \wedge \varphi$.

□

Even though, lifting to the original signature fulfils certain properties that correspond to the notions of focussing and the fundamental ideas of forgetting, we were able to show that after marginalizing and lifting an OCF it is possible to infer non-trivial conditionals mentioning the forgotten signature elements. These conditionals could not be inferred by the initial OCF κ. For this let κ be an OCF over signature Σ_{Tweety}, $\kappa_{|\Sigma_{Tweety}\setminus\{b\}}$ its marginalization, and $(\kappa_{|\Sigma_{Tweety}\setminus\{b\}})_{\uparrow\Sigma_{Tweety}}$ the lifting of the previously marginalized OCF back to the original signature as given in Tab. 14 below. The resulting OCF $(\kappa_{|\Sigma_{Tweety}\setminus\{b\}})_{\uparrow\Sigma_{Tweety}}$ is capable of inferring $(b|pbf \vee \overline{p}\overline{b}f)$, which contains the forgotten signature element b and is non-trivial in the sense that $[\![pbf \vee \overline{p}\overline{b}f]\!]_\Sigma \nsubseteq [\![b]\!]_\Sigma$ and $b \notin Bel((\kappa_{|\Sigma_{Tweety}\setminus\{b\}})_{\uparrow\Sigma_{Tweety}})$. On the contrary, the initial OCF κ is not capable of inferring the same conditional.

$$\begin{aligned}
&\kappa \models (b|pbf \vee \overline{p}\overline{b}f) \\
\Leftrightarrow\ &\kappa(pbf) < \kappa(\overline{p}\overline{b}f) \\
\Leftrightarrow\ &2 < 1 \quad \lightning
\end{aligned}$$

The reason why new inferences became possible after lifting is the changed order of the interpretations compared to the initial belief state, in which not only interpretations with different ranks are now assigned to the same rank, but also interpretations that were less plausible than others are more plausible afterwards, as illustrated in Tab. 14 above. There, it can be seen that $\overline{p}\overline{b}f \prec_\kappa pbf$ holds in the original OCF κ, while the $pbf \prec_{(\kappa_{|\Sigma_{Tweety}\setminus\{b\}})_{\uparrow\Sigma_{Tweety}}} \overline{p}\overline{b}f$ holds after marginalizing and lifting κ. In general, the changed order occurs when there already exist interpretations for which all other interpretations that are equivalent in the reduced

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	pbf, $pb\overline{f}$
1	$\overline{p}bf$, $\overline{p}b\overline{f}$, $\overline{p}\overline{b}f$, $\overline{p}\overline{b}\overline{f}$
0	$p\overline{b}f$, $p\overline{b}\overline{f}$

$\kappa_{\vert\Sigma_{Tweety}\setminus\{b\}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}\setminus\{b\}}$
∞	-
$\vdots$	-
3	-
2	-
1	$\overline{p}f$, $\overline{p}\overline{f}$
0	pf, $p\overline{f}$

$(\kappa_{\vert\Sigma_{Tweety}\setminus\{b\}})_{\uparrow\Sigma_{Tweety}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	-
1	$\overline{p}bf$, $\overline{p}b\overline{f}$, $\overline{p}\overline{b}f$, $\overline{p}\overline{b}\overline{f}$
0	pbf, $pb\overline{f}$, $p\overline{b}f$, $p\overline{b}\overline{f}$

Table 14: Top left: OCF κ over signature Σ_{Tweety}. Top right: Marginalization of κ to the reduced signature $\Sigma_{Tweety} \setminus \{b\}$. Bottom: Lifting the marginalized OCF $\kappa_{|\Sigma_{Tweety}\setminus\{b\}}$ to the original signature Σ_{Tweety}.

signature are assigned to the same rank (see $\kappa(\omega) = 1$ in Tab. 14). The ranks of those interpretations are not affected by the marginalization and lifting. This makes it possible for other interpretations to become more plausible, since the rank will be reduced to the minimum rank among all interpretations that are equivalent in the reduced signature. In the given example, the interpretations pbf and $pb\overline{f}$ were assigned to rank 2, and therefore were less plausible than the interpretations with rank 1. After marginalizing and lifting the OCF, pbf and $pb\overline{f}$ were assigned to rank 0, and therefore became more plausible than the interpretations assigned to rank 1. We want to register this observation in Obs. 3.34.

Observation 3.34. *Let κ be an OCF over signature Σ and $\Sigma' \subseteq \Sigma$ a subsignature. When we first marginalize κ to Σ', and lift the resulting OCF $\kappa_{|\Sigma'}$ back to Σ, then there exist non-trivial conditionals $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ with*

$$\kappa \not\models (\psi|\varphi) \text{ and } (\kappa_{|\Sigma'})_{\uparrow\Sigma} \models (\psi|\varphi).$$

This raises the question whether a first marginalized and then lifted OCF $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ can still be regarded as focussing on Σ', and furthermore as forgetting $\Sigma \setminus \Sigma'$ or not. As long as we only consider the belief set, $(\kappa_{|\Sigma'})_{\uparrow\Sigma}$ clearly can be considered as focussing on Σ', since all inferences over Σ' remain unchanged and only trivial inferences over the forgotten elements are possible (Prop. 3.32). But when we also consider conditionals, we are able to infer new non-trivial conditionals that could not be inferred by the initial belief state. This contradicts the general notion of forgetting, that the knowledge should be reduced without gaining any new information. However, there are a few aspects worth mentioning at this point. First, even though new conditionals could be inferred after marginalizing and lifting an OCF, the belief sets are only expanded by tautologies and formulas that can be inferred by the beliefs of the marginalized OCF. Thus, both belief sets are equivalent with respect to the original signature Σ. Moreover, we know that the minimization of changes in conditional beliefs is not always worth striving for, and that in most cases there do not exist optimal solutions, since preserving conditional beliefs often results in greater changes in propositional beliefs and vice-versa [DP97, Bou93]. Nonetheless, some properties of lifting the marginalized belief state to the original signature reflect the notions of focussing, such as the preservation of the conditionals over the reduced signature and the invariant behaviour towards the extension of their antecedences.

3.2.2 Contraction

When it comes to the principle of forgetting, different kinds like forgetting certain information over time or rejecting present knowledge due to new insights come to our minds. Many of those kinds were described by Kern-Isberner et al. in [BKIS$^+$19], but probably the most obvious kind is the direct forgetting of certain beliefs about objects of our world – the contraction. Everyday life examples for contractions could be forgetting where we parked the car, when our train arrives or whether penguins can fly. The most prominent and widely agreed on understanding of contractions was presented by Alchourrón, Gärdenfors and Makinson (AGM) be means of the AGM postulates [AGM85], which describe general properties a contraction should fulfil (see Section 2.3). They describe among others the property that after the contraction, we are no longer able to infer the just contracted formula, or that due to the contraction no new knowledge is generated. Since those postulates as originally stated by AGM only argue about knowledge as sets of formulas, which is not always an appropriate representation, they were generalized to arbitrary epistemic states by Konieczny and Pérez in [KP17]. In the further course, we will focus our research on c-contractions as presented as a kind of forgetting in [BKIS$^+$19] and originally stated in [KIBSB17], since they generalize the concept of contractions to epistemic states and conditionals, and therefore can be used in a much more expressive framework.

Before we actually start the elaboration, we want to note that according to Delgrande, contractions should not be considered as forgetting, since they are conceptually different [Del17]. The notion of forgetting described by Delgrande, exactly follows the definition of their general forgetting approach (Def. 3.1). Thus, forgetting, according to Delgrande, always means forgetting a certain object or concept (signature element) of our world, such that we are no longer able to argue about it. Taking up the small everyday life example from above again, this would mean that we would forget about the ability to fly and penguins in general, instead of just forgetting the simple fact that penguins cannot fly. Thus, we agree when they argue that contractions are conceptually different from their understanding of forgetting. But, even if Delgrande's general approach is capable of expressing several other logic specific forgetting approaches, such as Boole's forgetting in propositional logic [Boo54] (Th. 3.22), this clearly illustrates that Delgrande should not claim the term of *forgetting* for their approach. From a commonsense and psychological perspective, forgetting a certain fact can clearly be considered as a kind of forgetting, just as forgetting about objects or whole concepts of our world can. Thus, we believe that the concept of forgetting is much wider than described by Delgrande, which is why we disagree with them and consider their general approach just as one kind of forgetting, instead of as *the* general approach. Nonetheless, we appreciate the work of Delgrande, since the there presented insights are another important building block towards the understanding of what forgetting actually is.

Following the AGM contraction postulates as originally presented in [AGM85], contractions are always applied to knowledge sets and propositions. As mentioned above, this is not always suitable, since knowledge is most often not just represented by sets of formulas, but more generally by epistemic states, which enrich our beliefs by additional information like a quantitative or qualitative ranking of our knowledge. Thus, applying a contraction to prior knowledge not only affects our beliefs, but also the way our knowledge is organized. Therefore, it is of particular interest how previously performed belief changes affect the further. Applying multiple belief changes consecutively is also known as iterated belief change. In this context, Kern-Isberner et al. developed c-contractions, which are belief change operators contracting conditionals or propositions from epistemic states [KIBSB17]. These belong to a more general family of belief change operators called c-changes (Def. 3.35), which describe a general scheme for belief change operators satisfying the fundamental principle of conditional preservation. The principle of conditional preservation [Ker18] can be viewed as a conditional counter part to the minimal change paradigm, since it guarantees that conditional relations will not be rejected due to a belief change, if there is no need to reject them.

Definition 3.35. [KIBSB17] *Let κ be an OCF over signature Σ, $(\psi|\varphi) \in (\mathcal{L}_\Sigma|\mathcal{L}_\Sigma)$ a conditional, and $\circ$ a belief change operator. $\circ$ fulfils the* principle of conditional preservation *and is called a* c-change*, if there exist $\gamma^+, \gamma^-, \kappa_0 \in \mathbb{Z}$ such that*

$$\kappa \circ (\psi|\varphi)(\omega) = \kappa(\omega) - \kappa_0 + \begin{cases} \gamma^+, & \textit{if } \omega \models \varphi \wedge \psi \\ \gamma^-, & \textit{if } \omega \models \varphi \wedge \neg\psi \\ 0, & \textit{if } \omega \models \neg\varphi \end{cases}$$

A c-change $\kappa \circ (\psi|\varphi)$ divides all interpretations ω into three classes, depending on whether ω verifies or falsifies $(\psi|\varphi)$, or is not applicable at all. The latter interpretations are neutral towards $(\psi|\varphi)$, since they neither verify nor falsify $(\psi|\varphi)$. Thus, their plausibility is not affected by the belief change, which corresponds to changing their ranks by 0. For the remaining interpretations, we change their ranks by a constant factor γ^+ or γ^-, respectively. This way, the ranks of all interpretations that behave the same towards $(\psi|\varphi)$ are changed in the same manner, and further unmotivated changes are prevented. κ_0 works as a normalization constant maintaining the conditions necessary for an OCF, i.e. all ranks are positive and there exist interpretations with rank 0. C-changes can also be applied in a purely propositional framework. In this case the interpretations are only separated into two classes. An interpretation either satisfies or falsifies a formula. Thus, the third case of non-applicability is omitted. The actual effect of a c-change is then determined by further restrictions on the parameters γ^+, γ^- and κ_0. Thus, by means of the definition of c-changes, we can define c-contractions as follows (Def. 3.36).

Definition 3.36. [KIBSB17] *A c-change $\ominus$ is called* c-contraction*, if and only if for $\kappa \ominus (\psi|\varphi)$ the parameters $\gamma^-, \gamma^+, \kappa_0 \in \mathbb{Z}$ fulfil the following constraints:*

$$\begin{aligned} \kappa_0 &= \min\{\gamma^- + \kappa(\varphi \wedge \neg\psi), \kappa(\neg\varphi)\} \\ \gamma^- - \gamma^+ &\leq \kappa(\varphi \wedge \psi) - \kappa(\varphi \wedge \neg\psi). \end{aligned}$$

The parameters $\gamma^-, \gamma^+, \kappa_0$ originate from the definition of c-changes (Def. 3.35).

The further restrictions stated in Def. 3.36 can be directly derived from the c-contraction's underlying success postulate, which says that after the contraction of a certain conditional, we will not be able to infer it anymore, i.e. $\kappa \ominus (\psi|\varphi) \not\models (\psi|\varphi)$. The restrictions on γ^+ and γ^- are equivalent to the contraction's success postulate:

$$\begin{aligned} & \kappa \ominus (\psi|\varphi) \not\models (\psi|\varphi) \\ \Leftrightarrow\ & \kappa \ominus (\psi|\varphi)\ (\neg\psi|\varphi) \leq \kappa \ominus (\psi|\varphi)\ (\psi|\varphi) && \text{(Lem. 2.48)} \\ \Leftrightarrow\ & \kappa(\neg\psi|\varphi) - \kappa_0 + \gamma^- \leq \kappa(\psi|\varphi) - \kappa_0 + \gamma^+ && \text{(Def. 3.36)} \\ \Leftrightarrow\ & \kappa(\neg\psi|\varphi) + \gamma^- \leq \kappa(\psi|\varphi) + \gamma^+ \\ \Leftrightarrow\ & \kappa(\varphi \wedge \neg\psi) - \kappa(\varphi) + \gamma^- \leq \kappa(\varphi \wedge \psi) - \kappa(\varphi) + \gamma^+ && \text{(Def. 2.46)} \\ \Leftrightarrow\ & \kappa(\varphi \wedge \neg\psi) + \gamma^- \leq \kappa(\varphi \wedge \psi) + \gamma^+ \\ \Leftrightarrow\ & \gamma^- - \gamma^+ \leq \kappa(\varphi \wedge \psi) - \kappa(\varphi \wedge \neg\psi). \end{aligned}$$

When shifting the ranks by γ^+ or γ^-, we know that there are two cases in which the ranks must be adjusted by the normalization constant κ_0 in order to maintain the OCF's conditions. In the first case, the rank of $\varphi \wedge \neg\psi$ becomes negative when shifting by γ^-. Therefore, all ranks must be increased by $\kappa_0 = \gamma^- + \kappa(\varphi \wedge \neg\psi)$, such that they are greater or equal to 0 in the posterior OCF. In the other case, when increasing the rank of $\varphi \wedge \psi$ it can happen that no more interpretations are assigned to rank 0 afterwards. Thus, we have to decrease all ranks by either $\kappa_0 = \gamma^- + \kappa(\varphi \wedge \neg\psi)$ or $\kappa_0 = \kappa(\neg\varphi)$, whichever is smaller. This way it is guaranteed that there exist interpretations with rank 0 in the posterior OCF. In Ex. 3.3 we illustrate a c-contraction and how the parameter choice guarantees the success of the contraction.

Example 3.3. *In this example we illustrate how c-contractions affect a prior OCF and how the parameter restrictions guarantee the fulfilment of the corresponding success postulate. For this let κ be the OCF given in Tab. 15 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\overline{f}$
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$

$\kappa \ominus (\neg f\|p)\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	-
3	$\overline{p}b\overline{f}$, $p\overline{b}\overline{f}$
2	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$, $pb\overline{f}$, $p\overline{b}f$
1	-
0	pbf

Table 15: Left: OCF κ over signature Σ_{Tweety}. Right: Result of contracting $(\neg f|p)$ in κ with parameters $\gamma^- = -4$, $\gamma^+ = -1$ and $\kappa_0 = -2$.

If we now want to contract that penguins usually cannot fly $(\neg f|p)$ from κ, the parameters γ^+, γ^- and κ_0 must fulfil the following restrictions:

$$\gamma^- - \gamma^+ \leq \kappa(p \wedge \neg f) - \kappa(p \wedge f) = 1 - 2 = -1$$
$$\kappa_0 = \min\{\gamma^- + \kappa(p \wedge f), \kappa(\neg p)\} = \min\{\gamma^- + 2, 0\}$$

According to these restrictions, we choose $\gamma^- = -4$, $\gamma^+ = -1$ and $\kappa_0 = -2$. The choice of the parameters in this example is rather arbitrary and does not follow any particular strategy. Contracting $(\neg f|p)$ with the parameters as chosen above results in the posterior OCF $\kappa \ominus (\neg f|p)$, which is also given in Tab. 15. Due to the contraction, the ranks of all models of $p \wedge \neg f$ are increased by $-\kappa_0 + \gamma^+ = 1$, while the ranks of all models of $p \wedge f$ are shifted by $-\kappa_0 + \gamma^- = -2$. The remaining interpretations that are not applicable since they satisfy $\neg p$, are only shifted by the normalization constant $-\kappa_0 = 2$. This way all interpretations that behave equivalently towards $(\neg f|p)$ are treated the same by the c-contraction. Since the parameter restrictions directly derive from the underlying success postulate, it is guaranteed that the minimal models of $\neg f \wedge p$ are no more plausible than the minimal models of $f \wedge p$ after the contraction. Therefore, we are no longer able to infer $(\neg f|p)$ afterwards.

$$\begin{aligned} & \kappa \ominus (\neg f|p) \models (\neg f|p) & \\ \Leftrightarrow\ & \kappa \ominus (\neg f|p)\ (\neg f \wedge p) < \kappa \ominus (\neg f|p)\ (f \wedge p) & \text{(Prop. 2.49)} \\ \Leftrightarrow\ & 2 < 0 & \lightning \end{aligned}$$

Furthermore, it is worth mentioning that due to the c-contraction the prior beliefs have changed completely, since none of the prior most plausible interpretations were maintained.

As seen in Ex. 3.3 above, c-contractions, even if called *contractions*, do not generally satisfy the AGM postulates for epistemic states **(AGMes-1)**-**(AGMes-6)** (see Section 2.3 or Appendix A.1), since it is possible to induce almost arbitrary changes to the prior beliefs, as long as the contracted conditional cannot be inferred by the posterior OCF. Moreover, it would be necessary to consider propositional c-contractions only, since the AGM postulates are of a purely propositional nature. This can be traced back to the fact, that c-contractions are based on the principle of conditional preservation, while AGM theory is based on the minimal change paradigm. Moreover, this makes it difficult to argue about the changes induced by a c-contraction in general, and therefore to argue about properties of c-contractions. However, this problem can be solved by considering c-contractions of propositions that only induce minimal changes to the prior beliefs. We refer to them as minimal change c-contractions (Def. 3.37).

Definition 3.37. [KIBSB17] *Let κ be an OCF over signature Σ and $\varphi \in \mathcal{L}_\Sigma$ a formula. A* minimal change c-contraction $\kappa \ominus \varphi$ *is a propositional c-contraction with $\gamma^+ = 0$, $\gamma^- = \min\{0, \kappa(\varphi) - \kappa(\neg\varphi)\}$, and $\kappa_0 = \gamma^- + \kappa(\neg\varphi)$, where $\gamma^-, \gamma^+, \kappa_0$ are the parameters originating from the definition of c-changes (Def. 3.35).*

Note that Kern-Isberner et al. refer to minimal change c-contractions as propositional minimal type α c-contractions [KIBSB17]. But since neither the differentiation between type α and type β c-contractions, nor conditional type α c-contractions are necessary for our examinations, we only refer to them as minimal change c-contractions. Given this definition, we see that only the ranks of those interpretations falsifying φ are changed. By choosing γ^- as the minimum of 0 and $\kappa(\varphi) - \kappa(\neg\varphi)$, we know that the ranks of the models of $\neg\varphi$ are only shifted by the minimum amount necessary in order to guarantee that φ cannot be inferred by the posterior beliefs. Concretely, this means that if φ could be inferred by the prior beliefs, γ^- will be chosen as $-\kappa(\neg\varphi)$, since we know that $\kappa \models \varphi$ holds only if $\kappa(\varphi) = 0$. This way, both φ and $\neg\varphi$ are assigned to rank 0 by the posterior OCF, and φ can no longer be inferred. If φ could not be inferred by the prior beliefs, we choose $\gamma^- = 0$, which means that no ranks will be changed at all, since $\gamma^+ = 0$ holds as well. Therefore, following Def. 3.37, the posterior rank assignment can be described as given in Lem. 3.38.

Lemma 3.38. *Let κ be an OCF over signature Σ, $\varphi \in \mathcal{L}_\Sigma$ a formula, and $\ominus$ a minimal change c-contraction, then the following holds for all $\omega \in \Omega_\Sigma$:*

$$\text{If } \kappa(\varphi) = 0, \text{ then } \kappa \ominus \varphi\ (\omega) = \kappa(\omega) + \begin{cases} 0, & \text{if } \omega \models \varphi \\ -\kappa(\neg\varphi), & \text{if } \omega \not\models \varphi \end{cases}$$

$$\text{If } \kappa(\varphi) > 0, \text{ then } \kappa \ominus \varphi\ (\omega) = \kappa(\omega)$$

Kern-Isberner et al. elaborated in [KIBSB17] that the form of a propositional c-contraction determines, whether it satisfies **(AGMes-1)-(AGMes-7)** (Th. 3.39).

Theorem 3.39. [KIBSB17] *A propositional c-contraction $\ominus$ satisfies **(AGMes-1)-(AGMes-7)**, if and only if it has the form*

$$\kappa \ominus \varphi\ (\omega) = \kappa(\omega) + \begin{cases} \gamma^+ - \gamma^-, & \text{if } \omega \models \varphi \\ 0, & \text{if } \omega \not\models \varphi \end{cases}$$

for all $\varphi \in \mathcal{L}_\Sigma$ *with* $\kappa(\varphi) > 0$*, and*

$$\kappa \ominus \varphi\ (\omega) = \kappa(\omega) + \begin{cases} 0, & \textit{if } \omega \models \varphi \\ -\kappa(\neg\varphi), & \textit{if } \omega \not\models \varphi \end{cases}$$

for all $\varphi \in \mathcal{L}_\Sigma$ *with* $\kappa(\varphi) = 0$*.*

Minimal change c-contractions obviously match the form stated in Th. 3.39 above. For those $\varphi \in \mathcal{L}_\Sigma$ with $\kappa(\varphi) = 0$, the form in Lem. 3.38 exactly matches the form in Th. 3.39. For those $\varphi \in \mathcal{L}_\Sigma$ with $\kappa(\varphi) > 0$ a minimal change c-contraction chooses $\gamma^- = 0 = \gamma^+$, and therefore the form matches as well. In conclusion, we know that minimal change c-contractions satisfy **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40).

Proposition 3.40. *Let* $\ominus$ *be a minimum change c-contraction, then* $\ominus$ *satisfies* ***(AGMes-1)**-**(AGMes-7)***.

Finally, we want to state how the prior most plausible interpretations are changed due to minimal change c-contractions. This is of particular importance, since this directly describes how the corresponding posterior beliefs relate to the prior. Kern-Isberner et al. state that the posterior most plausible interpretations $[\![\kappa \ominus \varphi]\!]$ of a propositional c-contraction satisfying **(AGMes-1)**-**(AGMes-7)** is given by the unification of the prior most plausible interpretations $[\![\kappa]\!]$ and the minimal models falsifying φ (Prop. 3.41). In case that φ could not be inferred by the prior OCF in the first place, no changes are applied at all, since the minimal models of $\neg\varphi$ are already included in the most plausible interpretations $[\![\kappa]\!]$. This exactly corresponds to the idea of changing the prior OCF just as much as necessary in order to guarantee that φ can no longer be inferred.

Proposition 3.41. [KIBSB17] *Let* κ *be an OCF over signature* Σ*,* $\varphi \in \mathcal{L}_\Sigma$ *a formula, and* $\ominus$ *a minimal change c-contraction. The posterior most plausible interpretations of* $\kappa \ominus \varphi$ *are given by*

$$[\![\kappa \ominus \varphi]\!] = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\}.$$

Prop. 3.41 directly concludes from the relations of **(AGMes-1)**-**(AGMes-7)** and the underlying total preorder $\preceq_\kappa$ stated by Konieczny and Pérez in [KP17], and Caridroit et al. in [CKM17] (Th. 3.42).

Theorem 3.42. [KP17] *Let* Ψ *be an epistemic state and* $\varphi \in \mathcal{L}_\Sigma$ *a formula. An operator* $-$ *satisfies* ***(AGMes-1)**-**(AGMes-7)****, if and only if there exists a corresponding faithful preorder* $\preceq_\Psi$*, such that*

$$[\![\Psi - \varphi]\!] = [\![\Psi]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\Psi\}.$$

There they show that the AGM contraction postulates for epistemic states restrict the changes of the prior most plausible interpretations to the minimal models falsifying the contracted formula. However, Konieczny and Pérez formalize this relation in a more general manner, by just assuming an epistemic state Ψ with an underlying faithful preorder $\preceq_\Psi$. Therefore, the epistemic state does not necessarily have to be represented by an OCF. The following example Ex. 3.4 illustrates the changes of the most plausible interpretations due to a minimal change c-contraction.

Example 3.4. *In this example, we illustrate how minimal change c-contractions affect a prior OCF and how the parameter restrictions guarantee that the prior most plausible interpretations are only expanded by the minimal models falsifying the contracted formula. For this let κ be the OCF given in Tab. 16 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\,\overline{f}$
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$

$\kappa \ominus \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	-
3	$p\overline{b}f$
2	-
1	$\overline{p}b\overline{f}$, pbf, $p\overline{b}\,\overline{f}$
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$, $pb\overline{f}$

Table 16: Left: OCF κ over signature Σ_{Tweety}. Right: Result of contracting $\varphi \equiv \neg p$ in κ with parameters $\gamma^- = -1$, $\gamma^+ = 0$ and $\kappa_0 = 0$.

In the following, we contract $\varphi \equiv \neg p$ from κ by means of a minimal change c-contraction. Thus, the parameters γ^-, γ^+ and κ_0 must fulfil the following conditions:

$$\begin{aligned}
\gamma^+ &= 0 \\
\gamma^- &= \min\{0, \kappa(\neg p) - \kappa(p)\} = \min\{0, 0 - 1\} = -1 \\
\kappa_0 &= \gamma^- + \kappa(p) = -1 + 1 = 0
\end{aligned}$$

Other than for arbitrary c-contractions as seen in Ex. 3.3, the further restrictions for minimal change c-contractions already set the parameters to specific values, such that they are uniquely given. Thus, applying the minimal change c-contraction of $\neg p$ to κ results in $\kappa \ominus \varphi$ as given in Tab. 16 above. We see that due to the minimal change c-contraction only the ranks of those interpretations falsifying $\neg p$ are affected. Concretely, they are decreased by the minimum value necessary, such that p is assigned to rank 0 afterwards. Since we were able to infer $\neg p$ before, we know that both p and $\neg p$ are assigned to rank 0 by the posterior OCF, and therefore $\neg p$ can no longer be inferred. Moreover, we see that all of the prior most plausible interpretations are still assigned to rank 0 by the posterior OCF. In addition, the minimal models of φ were added:

$$\begin{aligned}
[\![\kappa \ominus \neg p]\!] &= \{\overline{p}\,\overline{b}\,\overline{f}, \overline{p}bf, \overline{p}\,\overline{b}f, pb\overline{f}\} \\
&= \{\overline{p}\,\overline{b}\,\overline{f}, \overline{p}bf, \overline{p}\,\overline{b}f\} \cup \{pb\overline{f}\} \\
&= [\![\kappa]\!] \cup \min\{[\![p]\!], \preceq_\kappa\}
\end{aligned}$$

In summary, we illustrated c-contractions as a kind of forgetting formulas in OCFs, and thereby disagreed with Delgrande's statement that contractions are conceptually different to forgetting [Del17]. Due to the almost arbitrary changes that can be induced by c-contractions, we further introduced minimal change c-contractions, which perform a contraction with respect to the minimal change paradigm, and therefore satisfy the AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-7)**. Note, that in contrast to arbitrary c-changes, it is necessary to restrict minimal change c-contraction to propositions, in order to satisfy the AGM postulates and further properties. Afterwards, we showed that due to the underlying total preorder of the OCF and the AGM postulates, the posterior most plausible interpretations after a minimal change c-contraction are given by the unification of the prior most plausible interpretations and the minimal models falsifying the contracted formula. This directly allows us to argue about the induced belief changes as well.

3.2.3 Revision

Besides the marginalization (Section 3.2.1) and the contraction (Section 3.2.2), Kern-Isberner et al. presented several different kinds of forgetting in [BKIS$^+$19]. The third and last we discuss in this work is the concept of revision. What distinguishes revisions from the remaining kinds of forgetting is the fact that they do not explicitly describe a kind of forgetting, since the underlying intention states the incorporation of new knowledge into presented beliefs. However, since it is not guaranteed that the new knowledge does not contradict any of our present beliefs, it might be necessary to reject some of them. Thus, the forgetting stated by revisions is of implicit nature. If we for example consider that we believe that penguins are able to fly, but then come to know that penguins are actually not able to fly, we have to forget our present beliefs about flying penguins in order to incorporate the new fact without inducing any contradictions. This illustrates that forgetting by means of a revision describes the forgetting of beliefs about the objects and concepts of our world, instead of the objects and concepts themselves, which again contradicts to Delgrande's understanding of forgetting, as already discussed in Section 3.2.2. Moreover, the example given above illustrates that the forgetting stated by revisions is of particular interest when it comes to intentional forgetting, which describes the act of actively and knowingly rejecting certain beliefs. Just as for contractions, the understanding of what revisions actually are is strongly influenced by AGM theory, in which the fundamental properties of revisions were postulated [AGM85, GR95]. Since a purely propositional framework as assumed by AGM is not always appropriate to argue about belief changes, the revision postulates as originally stated were later generalized to arbitrary epistemic states and extended by Darwiche and Pearl [DP97]. In this context, Kern-Isberner et al. discussed the concept of revision as a kind of forgetting in [BKIS$^+$19] by means of c-revisions [KI04]. In the following, we specify the definition of c-revisions, their relation to the established revision postulates, and state further properties that are necessary for the examinations in this work.

As already stated in Section 3.2.2, knowledge sets and a purely propositional framework are not always suitable for arguing about belief changes, especially when it comes to conditional beliefs and iterated belief changes. In this context, the principle of conditional preservation [Ker18] is of particular importance, since it states that conditional relations are not rejected by belief changes, if not necessary. In Def. 3.35, we already specified c-changes – a family of belief change operators that satisfy the principle of conditional preservation by definition. Just as c-contractions, c-revisions are defined by means of c-changes as well (Def. 3.43).

Definition 3.43. [KI04] *A c-change $\circledast$ is called* c-revision*, if and only if for $\kappa \circledast (\psi|\varphi)$ the parameters $\gamma^-, \gamma^+, \kappa_0 \in \mathbb{Z}$ fulfil the following constraints:*

$$\kappa_0 = \min\{\gamma^+ + \kappa(\varphi \wedge \psi), \kappa(\neg\varphi)\}$$
$$\gamma^- - \gamma^+ > \kappa(\varphi \wedge \psi) - \kappa(\varphi \wedge \neg\psi)$$

The parameters $\gamma^-, \gamma^+, \kappa_0$ originate from the definition of c-changes (Def. 3.35).

The further parameter restrictions stated in Def. 3.43 can directly be derived from, and therefore also reflect the underlying success postulate of c-revisions, namely that after revising κ with a conditional $(\psi|\varphi)$, we want to be able to infer $(\psi|\varphi)$, i.e. $\kappa \circledast (\psi|\varphi) \models (\psi|\varphi)$:

$$\begin{aligned}
& \kappa \circledast (\psi|\varphi) \models (\psi|\varphi) & \\
\Leftrightarrow\ & \kappa \circledast (\psi|\varphi)\ (\neg\psi|\varphi) > \kappa \circledast (\psi|\varphi)\ (\psi|\varphi) & \text{(Def. 2.32)}\\
\Leftrightarrow\ & \kappa(\neg\psi|\varphi) - \kappa_0 + \gamma^- > \kappa(\psi|\varphi) - \kappa_0 + \gamma^+ & \text{(Def. 3.43)}\\
\Leftrightarrow\ & \kappa(\varphi \wedge \neg\psi) - \kappa(\varphi) - \kappa_0 + \gamma^- > \kappa(\varphi \wedge \psi) - \kappa(\varphi) - \kappa_0 + \gamma^+ & \text{(Def. 2.46)}\\
\Leftrightarrow\ & \kappa(\varphi \wedge \neg\psi) + \gamma^- > \kappa(\varphi \wedge \psi) + \gamma^+ & \\
\Leftrightarrow\ & \gamma^- - \gamma^+ > \kappa(\varphi \wedge \psi) - \kappa(\varphi \wedge \neg\psi) &
\end{aligned}$$

Thus, when shifting all models of $(\psi|\varphi)$ by γ^+ and all models of $(\neg\psi|\varphi)$ by γ^-, we know that the most plausible models of $(\psi|\varphi)$ are more plausible than those of $(\neg\psi|\varphi)$, and therefore we are able to infer $(\psi|\varphi)$ afterwards. The normalization constant κ_0 guarantees that all ranks in the posterior OCF are positive, and that there exist interpretations assigned to rank 0. In case that there exist negative ranks, we know that it is sufficient to choose $\kappa_0 = \gamma^+ + \kappa(\varphi \wedge \psi)$, even though there might exist models of $\kappa(\varphi \wedge \neg\psi)$ with a negative rank as well. This is due to the fact, that $\varphi \wedge \psi$ is more plausible than $\varphi \wedge \neg\psi$ after shifting, as stated by the c-revision's success postulate. Thus, when there exist negative ranks due to the shifting, $\gamma^+ + \kappa(\varphi \wedge \psi)$ must be the lowest rank. In case that shifting the models does not result in negative ranks, it can still be possible that no interpretations are assigned to rank 0. Therefore, the ranks of all interpretations must be decreased by the minimum rank available, in order to maintain this condition. Since we know that due to the shifting $\varphi \wedge \psi$ is more plausible than $\varphi \wedge \neg\psi$, it is sufficient to choose κ_0 as the minimum of $\gamma^+ + \kappa(\varphi \wedge \psi)$ and $\kappa(\neg\varphi)$. In the following example Ex. 3.5, we want to illustrate that the parameter restrictions guarantee the inference of $(\psi|\varphi)$ by the posterior OCF.

Example 3.5. *This example illustrates how the parameter restrictions given in Def. 3.43 guarantee that after a c-revision $\kappa \circledast (\psi|\varphi)$ we are able to infer $(\psi|\varphi)$. For this we consider the prior OCF κ as given in Tab. 17 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$pb\overline{f}$, $p\overline{b}\,\overline{f}$
2	$p\overline{b}f$
1	$\overline{p}b\overline{f}$, pbf
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$

$\kappa \circledast (\neg f\|p)\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$p\overline{b}f$, $\overline{p}b\overline{f}$
1	pbf, $\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$
0	$pb\overline{f}$, $p\overline{b}\,\overline{f}$

Table 17: Left: OCF κ over signature Σ_{Tweety}. Right: Result of revising κ with $(\neg f|p)$, where $\gamma^- = -2$, $\gamma^+ = -5$ and $\kappa_0 = -2$.

In the following, we want to revise κ with the conditional $(\neg f|p)$. First, we want to state that $(\neg f|p)$ cannot be inferred by κ, which means that κ must actually be changed in order to be able to infer $(\neg f|p)$:

$$
\begin{aligned}
& \kappa \models (\neg f|p) & \\
\Leftrightarrow\ & \kappa(p \wedge \neg f) < \kappa(p \wedge f) & \text{(Prop. 2.49)} \\
\Leftrightarrow\ & 3 < 1 & \lightning
\end{aligned}
$$

Due to the parameter restrictions given by the definition of c-revisions, we know that $\kappa \circledast (\neg f|p)$ constrains γ^+, γ^- and κ_0 as follows:

$$
\begin{aligned}
\gamma^- - \gamma^+ &> \kappa(p \wedge \neg f) - \kappa(p \wedge f) = 3 - 1 = 2 \\
\kappa_0 &= \min\{\gamma^+ + \kappa(p \wedge \neg f), \kappa(\neg p)\} = \min\{\gamma^+ + 3, 0\}
\end{aligned}
$$

For this example, we choose $\gamma^+ = -5$, $\gamma^- = -2$ and $\kappa_0 = -2$. Revising κ with $(\neg f|p)$ and the parameters as mentioned above, we obtain $\kappa \circledast (\neg f|p)$ as given in Tab. 17 above. We see that all models of $\neg f \wedge p$ are shifted by $-\kappa_0 + \gamma^+ = -3$, while all models of $f \wedge p$ are shifted by $-\kappa_0 + \gamma^- = 0$. The remaining interpretations that are not applicable to $(\neg f|p)$ are only shifted by the normalization constant κ_0 in order to prevent negative ranks. Shifting the ranks this way, the resulting OCF is capable of inferring $(\neg f|p)$, since the minimal models of $\neg f \wedge p$ are now more plausible than those of $f \wedge p$:

$$
\begin{aligned}
& \kappa \circledast (\neg f|p) \models (\neg f|p) & \\
\Leftrightarrow\ & \kappa \circledast (\neg f|p)\ (p \wedge \neg f) < \kappa \circledast (\neg f|p)\ (p \wedge f) & \text{(Prop. 2.49)} \\
\Leftrightarrow\ & 0 < 1 & \checkmark
\end{aligned}
$$

Despite the fact that c-revisions can generally be applied to conditionals, we only focus on propositional c-revisions in the further course. We do so, since the general properties of forgetting we examine in this work only argue about a purely propositional framework. Therefore, elaborating c-revisions of conditionals in detail is not necessary. In Def. 3.44 we state the definition of propositional c-revisions, which corresponds to Def. 3.43, when assuming the $\varphi \equiv \top$ in $(\psi|\varphi)$.

Definition 3.44. [KIH17] *A c-revision $\kappa \circledast \varphi$ is called* propositional c-revision*, if and only if $\varphi \in \mathcal{L}$ is a formula and the parameters $\gamma^-, \gamma^+, \kappa_0 \in \mathbb{Z}$ fulfil the following constraints:*

$$\kappa_0 = \gamma^+ + \kappa(\varphi)$$
$$\gamma^- - \gamma^+ > \kappa(\varphi) - \kappa(\neg\varphi)$$

The parameters $\gamma^-, \gamma^+, \kappa_0$ originate from the definition of c-changes (Def. 3.35).

In the following, we want to elaborate the connections between the minimal models of a formula φ, the **(AGMes∗1)**-**(AGMes∗6)** (see Section 2.3 or Appendix A.1) revision postulates, and propositional c-revisions with φ. These connections are already common in the domain of knowledge representation and go back to the work of Darwiche and Pearl [DP97]. However, since they are often just implicitly assumed when working with revisions, we want to explicitly state and prove them at this point. Other than propositional c-contractions, different propositional c-revisions with the same formula and OCF are not able to result in different posterior beliefs, even though the parameters can be chosen freely with respect to the stated restrictions. In fact, it can be shown that the posterior most plausible interpretations after revising an OCF with a formula φ, always correspond to the minimal models of φ (Th. 3.45).

Theorem 3.45. *Let κ be an OCF over signature Σ and $\circledast$ a propositional c-revision, then*

$$[\![\kappa \circledast \varphi]\!] = \min\{[\![\varphi]\!], \preceq_\kappa\}$$

holds for each formula $\varphi \in \mathcal{L}_\Sigma$.

Proof of Th. 3.45. We prove the equation stated in Th. 3.45, by first showing that the posterior most plausible interpretations can be divided into two subsets. The first subset only contains models of φ, whereas the second subset only contains models of $\neg\varphi$.

$$
\begin{aligned}
&[\![\kappa \circledast \varphi]\!] \\
&= \{\omega \in \Omega_\Sigma \mid \kappa \circledast \varphi\ (\omega) = 0\} && \text{(Def. 2.30)} \\
&= \{\omega \in \Omega_\Sigma \mid \kappa(\omega) - \kappa_0 + \begin{cases} \gamma^+, & \text{if } \omega \models \varphi \\ \gamma^-, & \text{if } \omega \models \neg\varphi \end{cases} = 0\} && \text{(Def. 3.44)} \\
&= \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) - \kappa_0 + \gamma^+ = 0\} \cup \{\omega \in [\![\neg\varphi]\!] \mid \kappa(\omega) - \kappa_0 + \gamma^- = 0\}
\end{aligned}
$$

Next, we conclude that the second subset $\{\omega \in [\![\neg\varphi]\!] \mid \kappa(\omega) - \kappa_0 + \gamma^- = 0\}$ must be empty due to the c-revision's success postulate $\kappa \circledast \varphi \models \varphi$, which holds if and only if $\kappa \circledast \varphi(\varphi) < \kappa \circledast \varphi(\neg\varphi)$ (Def. 2.32). Thus, we know that no models of $\neg\varphi$ are assigned to rank 0 by $\kappa \circledast \varphi$.

$$[\![\kappa \circledast \varphi]\!] = \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) - \kappa_0 + \gamma^+ = 0\}$$

Furthermore, we can conclude that the models of φ that are assigned to rank 0 afterwards, must be the minimal models of φ in κ.

$$\begin{aligned}
[\![\kappa \circledast \varphi]\!] &= \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) - \kappa_0 + \gamma^+ = 0\} \\
&= \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = \kappa_0 - \gamma^+\} \\
&= \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = \gamma^+ + \kappa(\varphi) - \gamma^+\} && \text{(Def. 3.44)} \\
&= \{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = \kappa(\varphi)\} \\
&= \min\{[\![\varphi]\!], \preceq_\kappa\} && \text{(Def. 2.50)}
\end{aligned}$$

□

While it is necessary for propositional c-contractions to further restrict them to minimal change c-contractions in order to satisfy the AGM contraction postulates **(AGMes-1)-(AGMes-7)**, Th. 3.45 alone is sufficient for propositional c-revisions to satisfy the AGM revision postulates **(AGMes∗1)-(AGMes∗6)** (Prop. 3.47). The relation of minimal models and **(AGMes∗1)-(AGMes∗6)** goes back to the work of Katsuno and Mendelzon [KM91], in which it was admittedly described with respect to total preorders, but nevertheless the revision was still performed on a knowledge set represented as a single formula. Darwiche and Pearl [DP97] then generalized this relation to revisions of epistemic states (Th. 3.46).

Theorem 3.46. [DP97] *Let Ψ be an epistemic state and $\varphi \in \mathcal{L}_\Sigma$ a formula. A revision operator $*$ satisfies **(AGMes∗1)-(AGMes∗6)**, if and only if there exists a corresponding faithful total preorder $\preceq_\Psi$, such that*

$$[\![\Psi * \varphi]\!] = \min\{[\![\varphi]\!], \preceq_\Psi\}.$$

Proposition 3.47. *Let $\circledast$ be a propositional c-revision, then $\circledast$ satisfies **(AGMes∗1)-(AGMes∗6)**.*

When we argue about revisions in the context of epistemic states and iterated belief change, we know that in addition to the postulates stated by AGM, the principle of conditional preservation plays an essential role as well. In [Ker18] Kern-Isberner stated the principle of conditional preservation by means of a property that does not argue about revisions in particular, but is applicable to arbitrary belief change operators. However, when we assume the belief change operator in this generalized form to be a revision, then it exactly corresponds to the postulates **(DP1)-(DP4)** (see Section 2.3 or Appendix A.1) formulated by Darwiche and Pearl [DP97] in order to capture the same principle specifically for revisions. Since we already know from the definition of c-changes (Def. 3.35) that c-revisions fulfil the principle of conditional preservation, they must especially satisfy **(DP1)-(DP4)** in conclusion (Prop. 3.48). For a detailed explanation on the relation between **(DP1)-(DP4)** and the generalized form of the principle of conditional preservation, we refer to [Ker18].

Proposition 3.48. *Let $\circledast$ be a c-revision, then $\circledast$ satisfies* ***(DP1)-(DP4)****.*

In the following, we state some of the properties of c-revisions that are implied by **(AGMes∗1)-(AGMes∗6)**. First of all, from the **(Third identity)** of AGM theory (see Section 2.3), we know that an expansion with φ can also be expressed by means of a revision, in case that $\neg\varphi$ cannot be inferred by our present beliefs. When we consider the revision to be a c-revision, we cannot make use of the identities of AGM theory, since they do not apply to epistemic states. Thus, we instead refer to the corresponding **(Third equivalence)** as stated in Section 2.3. This translates the idea of the **(Third identity)** to the equivalence of belief sets, stating that the beliefs of a revision $\Psi * \varphi$ are equivalent to those of expanding the prior belief by φ, if they do not contradict φ. Notice that in contrast to the identity, this equivalence is not capable of defining an expansion by means of a given revision. We show in Prop. 3.49 that c-revisions satisfy the **(Third equivalence)** [6].

Proposition 3.49. *Let κ be an OCF over signature Σ and $\circledast$ a c-revision, then $\circledast$ satisfies the* ***(Third equivalence)***

$$\text{if } \kappa \not\models \neg\varphi, \text{ then } Bel(\kappa \circledast \varphi) \equiv Bel(\kappa) + \varphi$$

for each formula $\varphi \in \mathcal{L}_\Sigma$.

Proof of Prop. 3.49. In the following, we assume $\kappa \not\models \neg\varphi \Leftrightarrow \kappa(\varphi) = 0$ (Lem. 2.33).

$$\begin{aligned}
Bel(\kappa \circledast \varphi) &\equiv Th([\![\kappa \circledast \varphi]\!]) && \text{(Lem. 2.39)}\\
&\equiv Th(\min\{[\![\varphi]\!], \preceq_\kappa\}) && \text{(Th. 3.45)}\\
&\equiv Th(\{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = \kappa(\varphi)\}) && \text{(Def. 2.50)}\\
&\equiv Th(\{\omega \in [\![\varphi]\!] \mid \kappa(\omega) = 0\}) && (\kappa(\varphi) = 0)\\
&\equiv Th(\{\omega \in [\![\varphi]\!] \mid \omega \models Bel(\kappa)\}) && \text{(Prop. 2.37)}\\
&\equiv Th(\{\omega \in \Omega_\Sigma \mid \omega \models Bel(\kappa) \text{ and } \omega \models \varphi\}) &&\\
&\equiv Th(\{\omega \in \Omega_\Sigma \mid \omega \models Bel(\kappa) \cup \{\varphi\}\}) && \text{(Lem. 2.11)}\\
&\equiv Th([\![Bel(\kappa) \cup \{\varphi\}]\!]) && \text{(Def. 2.9)}\\
&\equiv Cn(Bel(\kappa) \cup \{\varphi\}) && \text{(Prop. 2.26)}\\
&\equiv Bel(\kappa) + \varphi && \text{(Th. 2.28)}
\end{aligned}$$

□

Furthermore, Prop. 3.49 concludes that the changes induced to the prior beliefs by a c-revision satisfy the AGM expansion postulates **(AGM+1)-(AGM+6)** (see Section 2.3 or Appendix A.1), and therefore describe an AGM expansion, if $\kappa(\varphi) = 0$.

Proposition 3.50. *Let $\circledast$ be a c-revision, then the change from $Bel(\kappa)$ to $Bel(\kappa \circledast \varphi)$ satisfies* ***(AGM+1)-(AGM+6)****, if $\kappa(\varphi) = 0$.*

[6] Note that Prop. 3.49 and Prop. 3.50 are also part of the lecture "Fortgeschrittene Themen der Wissensrepräsentation" by Kern-Isberner at TU Dortmund University.

At this point, we like to highlight two details of Prop. 3.50. Firstly, the condition under which c-revisions satisfy **(AGM+1)-(AGM+6)**, namely $\kappa(\varphi) = 0$, exactly corresponds to the idea that $\neg\varphi$ must not be inferable by the prior epistemic state, since $\kappa(\varphi) = 0$ guarantees that $\neg\varphi$ cannot be inferred, even if $\kappa(\neg\varphi) = 0$ holds as well. Secondly, Prop. 3.50 argues about the extension postulates as originally stated by AGM, and not about a generalized form for epistemic states, even though $\circledast$ applies to OCFs. But since we only argue about the effect of c-revisions on the prior beliefs the postulates **(AGM+1)-(AGM+6)** are sufficient.

Knowing that a c-revision with φ represent a belief expansion in case that $\kappa(\varphi) = 0$, we further want to argue about the relation of c-revisions and the **(Levi identity)**, which states that a revision is defined by subsequently performing a contraction, rejecting any contradicting beliefs, and an expansion that actually adds the new knowledge. As already stated for the **(Third identity)** above, it is not possible to directly apply the **(Levi identity)** to epistemic states. Therefore, we refer to the corresponding **(Levi equivalence)** as stated in Section 2.3. This translates the idea of the **(Levi identity)** to the equivalence of belief sets, but does not define a revision by means of a contraction and an expansion. In Prop. 3.51, we show that c-revisions generally satisfy the **(Levi equivalence)**.

Proposition 3.51. *Let κ be an OCF over signature Σ, $-$ a belief change operator satisfying **(AGMes-1)-(AGMes-7)**, and $\circledast$ a c-revision, then $\circledast$ satisfies the **(Levi equivalence)***

$$Bel(\kappa \circledast \varphi) \equiv Bel(\kappa - \neg\varphi) + \varphi$$

for each $\varphi \in \mathcal{L}_\Sigma$.

Proof of Prop. 3.51. We know from Th. 3.42 that

$$[\![\Psi - \varphi]\!] = [\![\Psi]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\Psi\}$$

holds for any epistemic state Ψ with faithful ranking $\preceq_\Psi$, and belief change operator satisfying **(AGMes-1)-(AGMes-7)**. Since there exist a faithful ranking $\preceq_\kappa$ for each OCF κ (Prop. 2.43) we can further conclude

$$[\![\kappa - \neg\varphi]\!] = [\![\kappa]\!] \cup \min\{[\![\varphi]\!], \preceq_\kappa\}. \tag{3.2}$$

Eq. 3.2 especially states that after contracting $\neg\varphi$ from κ, φ is assigned to rank 0, i.e. $\kappa - \neg\varphi\ (\varphi) = 0$, since its minimal models were added to the most plausible interpretations. Thus, due to Prop. 3.50 we know that

$$Bel(\kappa - \neg\varphi) + \varphi \equiv Bel((\kappa - \neg\varphi) \circledast \varphi) \qquad \text{(Prop. 3.50)}$$

holds. Next, we show that the posterior most plausible interpretations must be the same, and therefore the posterior beliefs equivalent.

$$\begin{aligned}
& Bel(\kappa \circledast \varphi) \equiv Bel((\kappa - \neg\varphi) \circledast \varphi) & \\
\Leftrightarrow\ & [\![\kappa \circledast \varphi]\!] = [\![(\kappa - \neg\varphi) \circledast \varphi]\!] & \text{(Prop. 2.38)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_{\kappa - \neg\varphi}\} & \text{(Th. 3.45)}
\end{aligned}$$

As stated by Eq. 3.2, the minimal models of φ with respect to $\preceq_\kappa$ are assigned to rank 0 in the posterior OCF $\kappa - \neg\varphi$. Therefore, we know that there cannot exist any other models of φ with a lower rank. This concludes that the minimal models of φ are the same in both κ and $\kappa - \neg\varphi$. □

Furthermore, we can express an even stronger form of the **(Levi equivalence)** for c-revisions, since they also satisfy the **(Third equivalence)** (Prop. 3.49), in which we can generally replace the expansion by another c-revision (Prop. 3.52).

Proposition 3.52. *Let κ be an OCF over signature Σ, $\ominus$ a belief change operator satisfying **(AGMes-1)-(AGMes-7)**, and $\circledast$ a c-revision, then*

$$Bel(\kappa \circledast \varphi) \equiv Bel((\kappa \ominus \neg\varphi) \circledast \varphi)$$

holds for each $\varphi \in \mathcal{L}_\Sigma$.

This follows directly from Prop. 3.51 above. This way it is not necessary to define an extra expansion over epistemic states. Note that this is possible since the previously performed contraction guarantees the condition $\kappa(\varphi) = 0$, which is necessary for c-revisions to express an expansion as stated in the **(Third equivalence)**. In the following, we want to illustrate the equivalence stated in Prop. 3.52 with an example (Ex. 3.6).

Example 3.6. *This example illustrates the relation between c-revisions and contractions that satisfy **(AGMes-1)-(AGMes-7)** by means of the **(Levi equivalence)** and the **(Third equivalence)** as stated in Prop. 3.52. For this example we consider $\ominus$ to be a minimal change c-contraction, since we know that they satisfy the above-mentioned AGM postulates (Prop. 3.40). Furthermore, we consider the OCF κ over signature Σ_{Tweety} as given in Tab. 18 below. First, we revise κ with $\varphi \equiv p$. According to the definition of c-revisions (Def. 3.44), the following constraints must hold:*

$$\gamma^- - \gamma^+ > \kappa(\varphi) - \kappa(\neg\varphi) = 1 - 0 = 1$$
$$\kappa_0 = \gamma^+ + \kappa(\varphi) = \gamma^+ + 1$$

We freely choose $\gamma^- = 2$, $\gamma^+ = 0$ and $\kappa_0 = 1$, which results in $\kappa \circledast \varphi$ as given in Tab. 18. We see that all rank of the models of φ are decreased by 1, resulting in the minimal models of φ as the posterior most plausible interpretations. The ranks of the remaining interpretations are increased by 1.

Next, we show that when we first contract $\neg\varphi$ from κ by means of a minimal change c-contraction, and afterwards perform a revision with φ, we result in equivalent beliefs. For the minimal change c-contraction $\kappa_c^\circ = \kappa \ominus \neg\varphi$, we know that the parameters cannot be chosen, but are clearly given by definition. In this particular case, the parameters are $\gamma^- = -1$, $\gamma^+ = 0$ and $\kappa_0 = 0$. The resulting OCF is also given in Tab. 18. We see that due to the minimal change c-contraction only the models of φ were affected, such that φ is assigned to rank 0 afterwards. When we

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$pb\overline{f}$, $p\overline{b}\,\overline{f}$
2	$p\overline{b}f$
1	$\overline{p}b\overline{f}$, pbf
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$

$\kappa \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$\overline{p}b\overline{f}$, $pb\overline{f}$, $p\overline{b}\,\overline{f}$
1	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$, $p\overline{b}f$
0	pbf

$\kappa \ominus \neg\varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$pb\overline{f}$, $p\overline{b}\,\overline{f}$
1	$\overline{p}b\overline{f}$, $p\overline{b}f$
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$, pbf

$\kappa_c^\circ \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$\overline{p}b\overline{f}$, $pb\overline{f}$, $p\overline{b}\,\overline{f}$
1	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf$, $\overline{p}\,\overline{b}f$, $p\overline{b}f$
0	pbf

Table 18: Top left: OCF κ over signature Σ_{Tweety}. Top right: Result of revising κ with $\varphi \equiv p$, where $\gamma^- = 2$, $\gamma^+ = 0$ and $\kappa_0 = -1$. Bottom left: Result of the minimal change c-contracting $\kappa \ominus \neg p$. Bottom right: Result of revising $\kappa_c^\circ = \kappa \ominus \neg p$ with φ, where $\gamma^- = -1$, $\gamma^+ = 0$ and $\kappa_0 = 0$.

revise κ_c° with φ afterwards, the parameters must be chosen according to the following constraints:

$$\gamma^- - \gamma^+ > \kappa(\varphi) - \kappa(\neg\varphi) = 0 - 0 = 0$$
$$\kappa_0 = \gamma^+ + \kappa(\varphi) = \gamma^+ + 0 = \gamma^+$$

Thus, we freely choose $\gamma^- = -1$, $\gamma^+ = 0$ and $\kappa_0 = 0$ and result in $\kappa_c^\circ \ominus \varphi$ as stated in Tab. 18. Comparing the most plausible interpretations of $\kappa \ominus \varphi$ and $\kappa_c^\circ \ominus \varphi$, we see that the minimal models of φ with respect to $\preceq_\kappa$ form the only interpretations assigned to rank 0 in both posterior OCFs. This is guaranteed, since the contraction moves the minimal models of φ to rank 0 in order to prevent inferring $\neg\varphi$, and thus the following revision just removes the models of $\neg\varphi$ from rank 0, such that only the minimal models of φ remain. Note that in this example we not just result in equivalent beliefs, but also in identical posterior OCFs. However, this does not hold in general, but requires further assumptions on the parameters of the c-revision.

Finally, we want to state some properties of c-revisions that argue about the preservation of minimal models. These properties will be needed for the further examinations presented in this work. In Prop. 3.53, we state that revising an OCF κ with a formula φ preserves the minimal models of those formulas ψ, if its minimal models are included in those of φ.

Proposition 3.53. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas and $\circledast$ a propositional c-revision.*

$$\textit{If } \min\{[\![\psi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_\kappa\}, \textit{then } \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\}.$$

Proof of Prop. 3.53. We know from Th. 3.45, that after revising κ with φ the prior minimal models of φ form the posterior most plausible interpretations. Thus, we know that all prior minimal models of ψ must be included in the posterior most plausible interpretations as well, i.e.

$$\kappa \circledast \varphi\ (\omega) = 0, \text{ for all } \omega \in \min\{[\![\psi]\!], \preceq_\kappa\}.$$

Moreover, we can conclude from the above that the prior minimal models of ψ are the only models of ψ assigned to rank 0 after the revision. Therefore, all other models of ψ are assigned to a rank greater than 0. In conclusion, the prior and posterior minimal models of ψ must be equal. □

Intuitively, the equality stated in Prop. 3.53 holds, since all minimal models of ψ are included in those of φ, and therefore we know that the minimal models of ψ are assigned to rank 0 by the posterior OCF. If they are already assigned to rank 0 by the prior OCF, we know that they are not affected by the revision. Otherwise, we know that there cannot exist any other model of ψ that is assigned to an even lower rank. In both cases, we can conclude that the prior and posterior minimal models of ψ are equal.

In Prop. 3.54, we state that the revision of κ with ψ preserves the minimal models of all formulas φ that are more specific than ψ. This is due to the fact that c-revisions change the ranks of all interpretations that behave equivalently towards ψ in the same way. Therefore, the order of all models of $\varphi \models \psi$ is preserved, and so are the minimal models.

Proposition 3.54. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circledast$ a propositional c-revision.*

$$\textit{If } \varphi \models \psi, \textit{then } \min\{[\![\varphi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_{\kappa \circledast \psi}\}.$$

Proof of Prop. 3.54. Since all models of φ are models of ψ as well, we know that the posterior rank for each $\omega \in [\![\varphi]\!]$ after the revision $\kappa \circledast \psi$ is given by

$$\kappa \circledast \psi\ (\omega) = \kappa(\omega) - \kappa_0 + \gamma^+. \qquad \text{(Def. 3.44)}$$

Shifting the ranks of all $\omega \in [\![\varphi]\!]$ by the same constant factor $-\kappa_0 + \gamma^+$, we know that the order of those models does not change:

$$\kappa(\omega) < \kappa(\omega') \Leftrightarrow \kappa(\omega) - \kappa_0 + \gamma^+ < \kappa(\omega') - \kappa_0 + \gamma^+, \text{ for all } \omega, \omega' \in [\![\varphi]\!].$$

Therefore, we know that the prior and posterior minimal models of φ must be equal, if $\varphi \models \psi$. □

In summary, we presented the concept of revision as a kind of forgetting. For this, we argued that the forgetting aspect of revisions is of purely implicit nature, since the main notion of revisions is the incorporation of new knowledge into present beliefs. Nonetheless, incorporating new knowledge might require the rejection of some presented beliefs in order to prevent contradictions. Therefore, revisions can be understood as intentionally forgetting those beliefs that contradict the new knowledge. Since our work will focus on forgetting in epistemic states, concretely OCFs, we further examined c-revisions as a concrete implementation of the revision concept, which were presented as a kind of forgetting in [BKIS+19]. Just as c-contractions, c-revisions belong to the family of belief change operators determined by the definition of c-changes (Def. 3.35), and therefore satisfy the principle of conditional preservation. Since the **(DP1)**-**(DP4)** (see Section 2.3 or Appendix A.1) postulates describe the same principle, but specifically for revisions, we know that c-revisions satisfy them as well. Moreover, we showed that the posterior most plausible models of a c-revision with κ exactly correspond to the minimal models of φ. This is of particular importance, since it is essential for c-revisions to satisfy the well-established revision postulates for epistemic state **(AGMes$*$1)**-**(AGMes$*$6)** (see Section 2.3 or Appendix A.1). The fact that c-revisions satisfy the AGM postulates without further restrictions shows that they are also conform to the minimal change paradigm. Further, we showed that c-revision satisfy the **(Levi equivalence)** and the **(Third equivalence)** (see Section 2.3), which correspond to the eponymous identities of the AGM theory. Finally, we formulated some properties stating the preservation of minimal models under certain conditions.

4 Towards a General Framework for Kinds of Forgetting

The main goal of this section is the elaboration of general forgetting postulates for epistemic states. Kern-Isberner et al. already presented certain forgetting postulates in [BKIS+19], stating the success conditions of the different kinds of forgetting. However, the here elaborated attempt of postulating general properties of forgetting goes beyond that, and states further properties arguing about the behaviour of forgetting in epistemic states. For this we make use of the postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4) as presented by Delgrande in [Del17], and generalize them such that they extend to epistemic states. Furthermore, we elaborate the relations between them and the three kinds of forgetting from [BKIS+19] presented in Section 3, namely the marginalization, contraction and revision.

Since the marginalization describes forgetting in the sense of a signature reduction, we will first examine in Section 4.1 whether it is possible to express Delgrande's approach by means of the marginalization, and thus if the kinds of forgetting presented in [BKIS+19] also cover the notions of forgetting as stated by Delgrande. Additionally, we compare the model theoretical considerations of both approaches. After this, we present a generalized version of **(DFP-1)**-**(DFP-7)** arguing about forgetting signature elements in epistemic states, and examine whether the marginalization satisfies them. At this point, we will also state the particular importance of the marginalization for the generalized postulates.

In Section 4.2, we will once again generalize the postulates **(DFP-1)**-**(DFP-7)** to epistemic states, but this time we transfer the fundamental notions of these postulates such that they argue about the forgetting of formulas. Thus, we present two types of forgetting postulates for epistemic states. We will then examine the latter for contractions in detail. Thereby, we emphasize the particular importance of the minimal change paradigm and the refinement relation of OCFs. Lastly, we elaborate the relations between the generalized forgetting postulates and the established AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-7)** (see Section 2.3 or Appendix A.1).

In Section 4.3, we will first state an explicit representation for the implicit forgetting performed by a revision by means of contractions. Afterwards, we examine how revisions relate to the generalized forgetting postulates and the forgetting properties of contractions. Finally, we will also elaborate the relations between the generalized forgetting postulates and the AGM revision postulates for epistemic states **(AGMes$*$1)**-**(AGMes$*$6)**, as well as for the postulates for iterated belief revision **(DP1)**-**(DP4)** (see Section 2.3 or Appendix A.1).

Finally, in Section 4.4 we want to discuss that the here elaborated attempt of postulating general properties of forgetting formulas is only partly suitable for the different kinds of forgetting. We do so by highlighting some controversial properties that are implied by them. Furthermore, we examine if a belief change operator satisfying them can exist at all. Lastly, we will present adjustments to the postulates that prevent some undesired behaviour, and therefore show that even if the generalized postulates are not yet suitable, they form another step towards a general framework for kinds of forgetting.

4.1 Marginalization / Focussing

In this section, we will compare Delgrande's general forgetting approach [Del17] presented in Section 3.1 to the OCF marginalization [BKIS$^+$19] presented in Section 3.2.1. The comparison of the approaches is straight-forward since they both consider forgetting as a reduction of the signature elements. In the following, we first show that the beliefs of $\kappa_{|\Sigma'}$ are equivalent to the result of forgetting a subsignature $\Sigma \setminus \Sigma'$ from a set of formulas Γ, i.e. $\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')$. We will also discuss the relation between Γ and $Bel(\kappa)$ necessary for a meaningful comparison of the two approaches. Note that due to the fact that Delgrande's approach only considers formulas and deductive inferences, we will not consider conditionals in the comparison of the approaches. Afterwards, we translate Delgrande's forgetting postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4) such that they are applicable to OCFs and prove that the marginalization satisfies all of them. Moreover, we show that the marginalization is not the only operator that satisfies these postulates, but clearly is of particular importance, because each other operator must induce additional propositional or conditional changes to the prior OCF.

4.1.1 On the Equivalence of Marginalizations and Delgrande's Forgetting Approach

In order to show the equivalence of the approaches, we first define some pre-conditions that are crucial for a meaningful comparison. Unlike the marginalization (Def. 3.23), which realizes forgetting by reducing the signature of an OCF, and thus of the corresponding beliefs in particular, $\mathcal{F}(\Gamma, P)$ (Def. 3.1) defines a function that applies the forgetting to a set of formulas Γ. This set of formulas can be regarded as the knowledge base from which the belief set $Cn(\Gamma)$ can be inferred deductively. This states the initial situation $\mathcal{F}(\Gamma, P)$ is applied to. For a meaningful comparison, we want the marginalization to work on the same initial situation, meaning that the beliefs of κ should be equivalent to $Cn(\Gamma)$. Otherwise, we would apply forgetting to two different initial situations, which makes a comparison pointless, because we then cannot expect the results of forgetting to be the same. This requirement can be fulfilled by choosing κ such that the most plausible interpretations $[\![\kappa]\!]$ equal the models of Γ. Since Lem. 2.39 states $Bel(\kappa) \equiv Th([\![\kappa]\!])$, we can conclude that we then obtain all formulas that are satisfied by the models of Γ, and therefore guarantee the same initial situation for both forgetting approaches. Given these pre-conditions, we show the equivalence of Delgrande's approach and the OCF marginalization in Th. 4.1 with respect to the beliefs that can be inferred after forgetting certain signature elements.

Theorem 4.1. *Let $\Gamma \subseteq \mathcal{L}_\Sigma$ be a set of formulas and κ an OCF over signature Σ with $Bel(\kappa) \equiv \Gamma$, then*

$$\mathcal{F}(\Gamma, \Sigma \setminus \Sigma') \equiv Bel(\kappa_{|\Sigma'})$$

holds for each subsignature $\Sigma' \subseteq \Sigma$.

Proof of Th. 4.1.

$$\begin{aligned}
Bel(\kappa_{|\Sigma'}) &\equiv Th_{\Sigma'}(\llbracket \kappa_{|\Sigma'} \rrbracket) && \text{(Lem. 2.39)}\\
&\equiv Th_{\Sigma'}(\llbracket \kappa \rrbracket_{|\Sigma'}) && \text{(Prop. 3.24)}\\
&\equiv Th_{\Sigma'}(\llbracket \Gamma \rrbracket_{|\Sigma'}) && (Bel(\kappa) \equiv \Gamma)\\
&\equiv Th_{\Sigma'}(\llbracket \mathcal{F}(\Gamma, \Sigma \setminus \Sigma') \rrbracket) && \text{(Th. 3.17)}\\
&\equiv Cn_{\Sigma'}(\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')) && \text{(Lem. 2.23)}\\
&\equiv \mathcal{F}(\Gamma, \Sigma \setminus \Sigma') && \textbf{(DFP-3)}
\end{aligned}$$

□

Delgrande shows in [Del17] that the there presented forgetting approach is capable of expressing several of the hitherto logic-specific forgetting definitions. This includes among other constant forgetting in first order logic [Del17], forgetting in disjunctive logic programs [Del17] and Boole's forgetting in propositional logic [Del17, Boo54]. Due to the equivalence stated in Th. 4.1, we can conclude that the marginalization should be capable of expressing those logic specific approaches, too. However, further examinations are needed for this. Since Boole's forgetting in propositional logic is also addressed in this work, we want to state the semantic equivalence of its result to the beliefs of a marginalized OCF explicitly in Cor. 4.2. For further relations that conclude directly from Th. 4.1, we refer to [Del17].

Corollary 4.2. *Let κ be an OCF over signature Σ and $\varphi \in \mathcal{L}_\Sigma$ a formula with $Bel(\kappa) \equiv \varphi$, then*

$$forget(\varphi, \rho) \equiv Bel(\kappa_{|\Sigma \setminus \{\rho\}})$$

holds for each atom $\rho \in \Sigma$.

Besides the relations to the specific approaches, we can conclude that Delgrande's model theoretical considerations (Th. 3.17 and 3.18) hold for the marginalization, too. In fact, we can even show that they correspond to the definition of marginalization. In Th. 4.3, we state that the models of the beliefs of a marginalized OCF relate to the models of the prior beliefs analogously to the relations of the models that hold for Delgrande's approach (Th. 3.17). Thereby, we argue about the models with respect to both the original and the reduced signature. Note that the signature Σ in the index of $\llbracket \cdot \rrbracket_\Sigma$ denotes the signature of the regarded models, which can also be omitted if it is clearly given by the context (see Def. 2.9). This is not to be confused with the reduction $\llbracket \cdot \rrbracket_{|\Sigma}$ or extension $\llbracket \cdot \rrbracket_{\uparrow\Sigma}$ of a model set (see Def. 3.12).

Theorem 4.3. *Let κ be an OCF over signature Σ and $\Sigma' \subseteq \Sigma$, then the following equations hold:*

1. $\llbracket Bel(\kappa_{|\Sigma'}) \rrbracket_{\Sigma'} = (\llbracket Bel(\kappa) \rrbracket_\Sigma)_{|\Sigma'}$

2. $\llbracket Bel(\kappa_{|\Sigma'}) \rrbracket_\Sigma = ((\llbracket Bel(\kappa) \rrbracket_\Sigma)_{|\Sigma'})_{\uparrow\Sigma}$

Proof of Th. 4.3. First, we prove the first equation stated in Th. 4.3. $[\![Bel(\kappa_{|\Sigma'})]\!]_{\Sigma'} = ([\![Bel(\kappa)]\!]_{\Sigma})_{|\Sigma'}$ is identical to $[\![Bel(\kappa_{|\Sigma'})]\!] = [\![Bel(\kappa)]\!]_{|\Sigma'}$, because $[\![\cdot]\!]_{\Sigma}$ only explicitly denotes the signature the models are defined over. Such annotations can be omitted, if the signature is clearly given by the context. Since we know from Lem. 2.40 that $[\![Bel(\kappa)]\!] = [\![\kappa]\!]$ holds and already showed $[\![\kappa_{|\Sigma'}]\!] = [\![\kappa]\!]_{|\Sigma'}$ in Prop. 3.24, we know that the first equation must hold.

Next, we prove the second equation stated.

$$
\begin{aligned}
&(([\![Bel(\kappa)]\!]_{\Sigma})_{|\Sigma'})_{\uparrow\Sigma} \\
&= ([\![Bel(\kappa)]\!]_{|\Sigma'})_{\uparrow\Sigma} && ([\![Bel(\kappa)]\!]_{\Sigma} = [\![Bel(\kappa)]\!]) \\
&= ([\![\kappa]\!]_{|\Sigma'})_{\uparrow\Sigma} && (\text{Lem. } 2.40) \\
&= [\![\kappa_{|\Sigma'}]\!]_{\uparrow\Sigma} && (\text{Prop. } 3.24) \\
&= [\![Bel(\kappa_{|\Sigma'})]\!]_{\uparrow\Sigma} && (\text{Lem. } 2.40) \\
&= [\![Bel(\kappa_{|\Sigma'})]\!]_{\Sigma} && (\text{Lem. } 3.15)
\end{aligned}
$$

□

The equations in Th. 4.3 correspond to the definitions of marginalization (Def. 3.23) and lifting (Def. 3.28) respectively. The models of $Bel(\kappa_{|\Sigma'})$ are those interpretations that are assigned to rank 0 by $\kappa_{|\Sigma'}$. These interpretations correspond to the most plausible interpretations of κ when reducing them to Σ'. Furthermore, each expansion of the models of $Bel(\kappa_{|\Sigma'})$ to a signature Σ with $\Sigma' \subseteq \Sigma$ results in turn in models of $Bel(\kappa_{|\Sigma'})$ as well, since the formulas in $Bel(\kappa_{|\Sigma'})$ do not mention any of the added signature elements, and thus act invariantly towards their interpretations. Therefore, Delgrande's model theoretical considerations (Th. 3.17 and 3.18) correspond to those of the marginalization and lifting. In conclusion, the models of $Bel(\kappa_{|\Sigma'})$ can also be defined analogously to those of $\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')$ (Th. 3.18) as stated in Cor. 4.4 below.

Corollary 4.4. *Let κ be an OCF over signature Σ and $\Sigma' \subseteq \Sigma$.*

$$[\![Bel(\kappa_{|\Sigma'})]\!]_{\Sigma} = \{\omega \in \Omega_{\Sigma} \mid \textit{there exists } \omega' \in [\![Bel(\kappa)]\!]_{\Sigma} \textit{ with } \omega \equiv_{\Sigma\setminus\Sigma'} \omega'\}$$

4.1.2 Postulates for Forgetting Signatures in Epistemic States

After we have shown that the beliefs of a marginalized OCF $\kappa_{|\Sigma'}$ are equivalent to the result of Delgrande's forgetting $\mathcal{F}(\Gamma, \Sigma \setminus \Sigma')$ when applied to $\Gamma \equiv Bel(\kappa)$ (Th. 4.1), and furthermore showed that the corresponding model theoretical consideration also hold for the marginalization (Th. 4.3 and Cor. 4.4), we want to continue with an examination on which of Delgrande's forgetting postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4) are satisfied by the marginalization. Since Delgrande describes those properties with respect to $\mathcal{F}(\Gamma, P)$ (Def. 3.1), they are not suitable to state general properties of forgetting. Thus, we extend **(DFP-1)**-**(DFP-7)** in the following, such that they do not depend on the definition of any particular forgetting operator. Furthermore, we introduce them to epistemic states in order to apply them to more expressive semantic frameworks, and apply them to the marginalization of OCFs. Thus, we refer to this extension as the signature forgetting postulates for epistemic states **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. Note that the just mentioned postulates can also be found in Appendix A.1 for a quick and easy access. Let Ψ, Φ be epistemic states over the same signature Σ and $P, P', P_1, P_2 \subseteq \Sigma$ be subsignatures:

(DFPes-1)$_\Sigma$ $Bel(\Psi) \models Bel(\Psi \circ_f^\Sigma P)$

(DFPes-2)$_\Sigma$ If $Bel(\Psi) \models Bel(\Phi)$, then $Bel(\Psi \circ_f^\Sigma P) \models Bel(\Phi \circ_f^\Sigma P)$

(DFPes-3)$_\Sigma$ If $P' \subseteq P$, then $Bel((\Psi \circ_f^\Sigma P') \circ_f^\Sigma P) \equiv Bel(\Psi \circ_f^\Sigma P)$

(DFPes-4)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma (P_1 \cup P_2)) \equiv Bel(\Psi \circ_f^\Sigma P_1) \cap Bel(\Psi \circ_f^\Sigma P_2)$

(DFPes-5)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma (P_1 \cup P_2)) \equiv Bel((\Psi \circ_f^\Sigma P_1) \circ_f^\Sigma P_2)$

(DFPes-6)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma P) \equiv Bel((\Psi \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P}$

We refer to Section 3.1 for a detailed explanation of the postulates, since **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ capture the same underlying ideas as **(DFP-1)**-**(DFP-7)** and the extension to epistemic states is straightforward. However, there are a few points we want to emphasize in particular. Since the beliefs of an epistemic state are deductively closed by definition, it is not necessary to maintain **(DFP-3)**. Notice that due to omitting **(DFP-3)** the postulates **(DFP-4)**-**(DFP-7)** correspond to **(DFPes-3)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. Furthermore, we expressed the forgetting in the original signature $\mathcal{F}_O(\Gamma, P)$ in **(DFP-7)** as the beliefs after forgetting P and lifting the posterior epistemic state back to the original signature. The models of $\mathcal{F}_O(\Gamma, P)$ are equal to the models of forgetting P in Γ in the reduced signature lifted back to the original signature, i.e. $[\![\mathcal{F}(\Gamma, P)]\!]_{\uparrow\Sigma}$ (Cor. 3.19). When we consider the models of $Bel((\Psi \circ_f^\Sigma P)_{\uparrow\Sigma})$, i.e. $[\![\Psi \circ_f^\Sigma P]\!]_{\uparrow\Sigma}$, we see that this also describes the models after forgetting P lifted back to the original signature. Therefore, **(DFPes-6)**$_\Sigma$ matches the property originally stated in **(DFP-7)**.

Finally, we prove that the marginalization satisfies all of the signature forgetting postulates **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. For this, we want to introduce an alternative notation of the marginalization by means of an operator $\circ_f^{\Sigma,m}$. This way, we can formulate the marginalization with a focus on which signature elements should be forgotten instead of which should be retained, and furthermore using $\circ_f^{\Sigma,m}$, the marginalization follows the notation given by **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. Thus, we refer to $\kappa_{|\Sigma \setminus P}$ as $\kappa \circ_f^{\Sigma,m} P$ in the further course. Moreover, we want to show that the beliefs of a marginalized OCF can be determined analogously to the result of Delgrande's forgetting approach (Def. 3.1) by means of intersecting with the reduced language in Prop. 4.5, since this will allow us to prove that the marginalization satisfies **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ more conveniently.

Proposition 4.5. *Let κ be an OCF over signature Σ and $\Sigma' \subseteq \Sigma$ a reduced signature.*

$$Bel(\kappa_{|\Sigma'}) = Bel(\kappa) \cap \mathcal{L}_{\Sigma'}$$

Proof of Prop. 4.5.

$$\begin{aligned} Bel(\kappa) \cap \mathcal{L}_{\Sigma'} &= Bel(\kappa_{|\Sigma'}) \cap \mathcal{L}_{\Sigma'} && \text{(Lem. 3.26)} \\ &= Bel(\kappa_{|\Sigma'}) && (Bel(\kappa_{|\Sigma'}) \subseteq \mathcal{L}_{\Sigma'}) \end{aligned}$$

□

Finally, we state in Th. 4.6 that the marginalization satisfies all of the abovementioned signature forgetting postulates **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**.

Theorem 4.6. *Let κ be an OCF over signature Σ. The marginalization $\kappa_{|\Sigma'}$ to a subsignature $\Sigma' \subseteq \Sigma$ satisfies **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**.*

Proof of Th. 4.6. In the following, we assume the epistemic states Ψ and Φ to be OCFs, since the marginalization is specifically defined over OCFs, and further denote the marginalization $\kappa_{|\Sigma \setminus P}$ as $\kappa \circ_f^{\Sigma,m} P$.

(DFPes-1)$_\Sigma$: Let $\Sigma' = \Sigma \setminus P$, then

$$\begin{aligned} & Bel(\kappa) \models Bel(\kappa \circ_f^{\Sigma,m} P) \\ \Leftrightarrow\ & \text{if } Bel(\kappa \circ_f^{\Sigma,m} P) \models \varphi, \text{ then } Bel(\kappa) \models \varphi, \text{ for all } \varphi \in \mathcal{L}_{\Sigma'} \\ \Leftrightarrow\ & \text{if } Bel(\kappa_{|\Sigma'}) \models \varphi, \text{ then } Bel(\kappa) \models \varphi, \text{ for all } \varphi \in \mathcal{L}_{\Sigma'} \end{aligned}$$

holds, since we already know from Lem. 3.26 that the beliefs of κ and $\kappa_{|\Sigma'}$ consist of the same formulas $\varphi \in \mathcal{L}_{\Sigma'}$.

(DFPes-2)$_\Sigma$: Let $\Sigma' = \Sigma \setminus P$, then

$$\begin{aligned} & Bel(\kappa \circ_f^{\Sigma,m} P) \models Bel(\kappa' \circ_f^{\Sigma,m} P) \\ \Leftrightarrow\ & Bel(\kappa_{|\Sigma'}) \models Bel(\kappa'_{|\Sigma'}) \\ \Leftrightarrow\ & Bel(\kappa) \cap \mathcal{L}_{\Sigma'} \models Bel(\kappa') \cap \mathcal{L}_{\Sigma'} && \text{(Prop. 4.5)} \\ \Leftrightarrow\ & Bel(\kappa') \cap \mathcal{L}_{\Sigma'} \subseteq Bel(\kappa) \cap \mathcal{L}_{\Sigma'} \end{aligned}$$

holds, since **(DFPes-2)$_\Sigma$** assumes $Bel(\kappa) \models Bel(\kappa')$, which is equivalent to $Bel(\kappa') \subseteq Bel(\kappa)$. Thus, when intersecting both belief sets with $\mathcal{L}_{\Sigma'}$ their subset relation is retained.

(DFPes-3)$_\Sigma$:

$$\begin{aligned} & Bel((\kappa \circ_f^{\Sigma,m} P') \circ_f^{\Sigma,m} P) \\ \equiv\ & Bel(\kappa_{\Sigma \setminus P'} \circ_f^{\Sigma,m} P) \\ \equiv\ & Bel((\kappa_{|\Sigma \setminus P'})_{|(\Sigma \setminus P') \setminus P}) \\ \equiv\ & Bel((\kappa_{|\Sigma \setminus P'})_{|(\Sigma \setminus (P' \cup P))}) \\ \equiv\ & Th(\llbracket (\kappa_{|\Sigma \setminus P'})_{|(\Sigma \setminus (P' \cup P))} \rrbracket) && \text{(Lem. 2.39)} \\ \equiv\ & \{\varphi \in \mathcal{L}_{\Sigma \setminus (P' \cup P)} \mid \llbracket (\kappa_{|\Sigma \setminus P'})_{|(\Sigma \setminus (P' \cup P))} \rrbracket \models \varphi\} && \text{(Def. 2.22)} \end{aligned}$$

$$
\begin{aligned}
&\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus(P'\cup P)} \mid [\![\kappa_{|\Sigma\setminus P'}]\!]_{|\Sigma\setminus(P'\cup P)} \models \varphi\} && \text{(Prop. 3.24)}\\
&\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus(P'\cup P)} \mid ([\![\kappa]\!]_{|\Sigma\setminus P'})_{|\Sigma\setminus(P'\cup P)} \models \varphi\} && \text{(Prop. 3.24)}\\
&\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus(P'\cup P)} \mid [\![\kappa]\!]_{|\Sigma\setminus(P'\cup P)} \models \varphi\} && \text{(Lem. 3.14)}\\
&\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus P} \mid [\![\kappa]\!]_{|\Sigma\setminus P} \models \varphi\} && (P' \subseteq P)\\
&\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus P} \mid [\![\kappa_{|\Sigma\setminus P}]\!] \models \varphi\} && \text{(Prop. 3.24)}\\
&\equiv Th([\![\kappa_{|\Sigma\setminus P}]\!]) && \text{(Def. 2.22)}\\
&\equiv Bel(\kappa_{|\Sigma\setminus P}) && \text{(Lem. 2.39)}\\
&\equiv Bel(\kappa \circ_f^{\Sigma,m} P)
\end{aligned}
$$

(DFPes-4)$_\Sigma$:

$$
\begin{aligned}
Bel(\kappa \circ_f^{\Sigma,m} (P_1 \cup P_2)) &\equiv Bel(\kappa_{|\Sigma\setminus(P_1\cup P_2)})\\
&\equiv Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} && \text{(Prop. 4.5)}\\
&\equiv Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}\\
&\equiv Bel(\kappa) \cap Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus P_2}\\
&\equiv (Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus P_1}) \cap (Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus P_2})\\
&\equiv Bel(\kappa_{|\Sigma\setminus P_1}) \cap Bel(\kappa_{|\Sigma\setminus P_2}) && \text{(Prop. 4.5)}\\
&\equiv Bel(\kappa \circ_f^{\Sigma,m} P_1) \cap Bel(\kappa \circ_f^{\Sigma,m} P_2)
\end{aligned}
$$

(DFPes-5)$_\Sigma$:

$$
\begin{aligned}
Bel(\kappa \circ_f^{\Sigma,m} P_1 \cup P_2) &\equiv Bel(\kappa_{|\Sigma\setminus(P_1\cup P_2)})\\
&\equiv Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} && \text{(Prop. 4.5)}\\
&\equiv Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup(P_1\cup P_2))}\\
&\equiv Bel(\kappa) \cap (\mathcal{L}_{\Sigma\setminus P_1} \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)})\\
&\equiv (Bel(\kappa) \cap \mathcal{L}_{\Sigma\setminus P_1}) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} && \text{(Associativity)}\\
&\equiv Bel(\kappa_{|\Sigma\setminus P_1}) \cap \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} && \text{(Prop. 4.5)}\\
&\equiv Bel(\kappa_{|\Sigma\setminus P_1}) \cap \mathcal{L}_{(\Sigma\setminus P_1)\setminus P_2}\\
&\equiv Bel((\kappa_{|\Sigma\setminus P_1})_{|(\Sigma\setminus P_1)\setminus P_2}) && \text{(Prop. 4.5)}\\
&\equiv Bel((\kappa \circ_f^{\Sigma,m} P_1)_{|(\Sigma\setminus P_1)\setminus P_2})\\
&\equiv Bel((\kappa \circ_f^{\Sigma,m} P_1) \circ_f^{\Sigma,m} P_2)
\end{aligned}
$$

(DFPes-6)$_\Sigma$:

$$
\begin{aligned}
Bel((\kappa \circ_f^{\Sigma,m} P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma'} &\equiv Bel((\kappa_{|\Sigma\setminus P})_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma'}\\
&\equiv Cn_\Sigma(Bel(\kappa_{|\Sigma\setminus P})) \cap \mathcal{L}_{\Sigma'} && \text{(Prop. 3.32)}\\
&\equiv Bel(\kappa_{|\Sigma\setminus P}) \cap \mathcal{L}_{\Sigma'} && \text{(Note)}\\
&\equiv Bel(\kappa_{|\Sigma\setminus P})\\
&\equiv Bel(\kappa \circ_f^{\Sigma,m} P)
\end{aligned}
$$

Note: Cn_Σ extends $Bel(\kappa_{|\Sigma'})$ by formulas mentioning elements of $\Sigma \setminus \Sigma'$, while the overall result does not change because the same formulas are removed by the intersection with $\mathcal{L}_{\Sigma'}$. Thus, we can omit Cn_Σ.

□

Since we showed in Th. 4.6 that the marginalization not only results in equivalent beliefs as Delgrande's forgetting approach (Th. 4.1), but also satisfies the signature forgetting postulates **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, we know in conclusion that the marginalization exactly captures Delgrande's idea of forgetting and extends it to OCFs.

In the further course, we want to illustrate that the marginalization is not the only operator satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, but clearly is of particular importance. For this, we first show that the posterior beliefs of $\kappa \circ_f^\Sigma P$ can always be inferred by the beliefs of the marginalized OCF $\kappa_{|\Sigma \setminus P}$, for each arbitrary operator $\circ_f^\Sigma$ satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ (Prop. 4.7).

Proposition 4.7. *Let κ be an OCF over signature Σ, $P \subseteq \Sigma$ a subsignature, and $\circ_f^\Sigma$ an operator satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, where $\kappa \circ_f^\Sigma P$ is an OCF over the reduced signature $\Sigma \setminus P$, then the following relation holds:*

$$Bel(\kappa_{|\Sigma \setminus P}) \models Bel(\kappa \circ_f^\Sigma P)$$

Proof of Prop. 4.7.

$$\begin{aligned}
& Bel(\kappa) \models Bel(\kappa \circ_f^\Sigma P) && \textbf{(DFPes-1)}_\Sigma \\
\Leftrightarrow\ & Bel(\kappa) \models Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} && \textbf{(DFPes-6)}_\Sigma \\
\Leftrightarrow\ & Bel(\kappa) \models \varphi, \text{ for all } \varphi \in Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} \subseteq \mathcal{L}_{\Sigma \setminus P} && \\
\Leftrightarrow\ & \kappa \models \varphi, \text{ for all } \varphi \in Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} \subseteq \mathcal{L}_{\Sigma \setminus P} && \text{(Prop. 2.41)} \\
\Leftrightarrow\ & \kappa_{|\Sigma \setminus P} \models \varphi, \text{ for all } \varphi \in Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} \subseteq \mathcal{L}_{\Sigma \setminus P} && \text{(Lem. 3.26)} \\
\Leftrightarrow\ & Bel(\kappa_{|\Sigma \setminus P}) \models \varphi, \text{ for all } \varphi \in Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} \subseteq \mathcal{L}_{\Sigma \setminus P} && \text{(Prop. 2.41)} \\
\Leftrightarrow\ & Bel(\kappa_{|\Sigma \setminus P}) \models Bel((\kappa \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P} && \\
\Leftrightarrow\ & Bel(\kappa_{|\Sigma \setminus P}) \models Bel(\kappa \circ_f^\Sigma P) && \textbf{(DFPes-6)}_\Sigma
\end{aligned}$$

□

From Prop. 4.7 and Prop. 3.25 we can conclude that the marginalization forms the operation satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ that induces minimal change to both the prior beliefs and the conditionals over the reduced signature that could be inferred by the prior OCF. This means that every other operator $\circ_f^\Sigma$ satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ removes additional formulas from the prior beliefs or changes the conditionals that could be inferred by the prior OCF, while possibly resulting in the same beliefs as the marginalization. Thus, the marginalization could also be regarded as the only pure signature reduction satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. At this point, we want to note that even though other operators satisfying these postulates exist, we are sceptical about their capability of describing meaningful cognitive procedures, since the additional changes they apply to the prior epistemic state are rather random. We illustrate two such signature forgetting operators in Ex. 4.1.

Example 4.1. *In the following, we give two examples for signature forgetting operators satisfying* ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$ *that are different from the marginalization. The first operator* $\circ_f^{\Sigma,1}$ *illustrates that it is possible for signature forgetting operators to satisfy* ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$*, if the posterior beliefs are always equivalent to those of the marginalization, while the order of all interpretations with a rank greater than 0 is changed every time. The second operator* $\circ_f^{\Sigma,2}$ *illustrates that it is even possible to satisfy* ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$*, if the most plausible interpretations extend those after the marginalization. In the following, we only intuitively explain why* $\circ_f^{\Sigma,1}$ *and* $\circ_f^{\Sigma,2}$ *satisfy* ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$*. For detailed proofs we refer to Appendix A.2.*

First, we examine $\circ_f^{\Sigma,1}$*, which is given by*

$$\kappa \circ_f^{\Sigma,1} P\ (\omega) = \begin{cases} 0, & \textit{if } \kappa_{|\Sigma\setminus P}(\omega) = 0 \\ \max\{\kappa_{|\Sigma\setminus P}\ (\omega) \mid \omega \in \Omega_{\Sigma\setminus P}\} - \kappa_{|\Sigma\setminus P}(\omega) + 1, & \textit{otherwise} \end{cases}.$$

Intuitively, this $\kappa \circ_f^{\Sigma,1} P$ *assigns the same interpretations to rank 0 as the marginalized OCF* $\kappa_{\Sigma\setminus P}$*, while inverting the order of the remaining interpretations (Tab. 19). Since the most plausible interpretations after reducing the signature with* $\circ_f^{\Sigma,1}$ *are always the same as for the marginalization, we further know* $Bel(\kappa_{|\Sigma\setminus P}) \equiv Bel(\kappa \circ_f^{\Sigma,1} P)$ *for all* $P \subseteq \Sigma$*. Moreover, since* ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$ *only argue about the beliefs of epistemic states, we know in conclusion that* $\circ_f^{\Sigma,1}$ *also satisfies them.*

Next, we give an example for a signature forgetting operator that does not always result in beliefs equivalent to those of the marginalization, but nonetheless satisfies ***(DFPes-1)***$_\Sigma$*-****(DFPes-6)***$_\Sigma$*. We refer to this operator as* $\circ_f^{\Sigma,2}$ *with*

$$\kappa \circ_f^{\Sigma,2} P\ (\omega) = \begin{cases} 0, & \textit{if } \omega \in \sigma(P)_{|\Sigma\setminus P} \\ \kappa_{|\Sigma\setminus P}(\omega), & \textit{otherwise} \end{cases},$$

where

$$\sigma(P) = \begin{cases} \bigcup_{\rho \in P} \sigma(\{\rho\}), & \textit{if } |P| > 1 \\ \{pbf\}, & \textit{if } P = \{p\} \\ \{pb\overline{f}\}, & \textit{if } P = \{b\} \\ \{p\overline{b}f\} & \textit{if } P = \{f\} \end{cases}$$

is a selection function that determines which interpretation should be added to rank 0, depending on the subsignature that should be forgotten. Since the ranks for all interpretations that are not selected by σ *equal the ranks assigned by the marginalized OCF, we know that the posterior most plausible interpretations of* $\kappa \circ_f^{\Sigma,2} P$ *must be equal to or extend those of the marginalized OCF* $\kappa_{|\Sigma\setminus P}$*, which matches the property described in Prop. 4.7. In the further course, we first illustrate the*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{p}\overline{b}f$, $\overline{p}\overline{b}\overline{f}$, $p\overline{b}\overline{f}$
2	pbf, $pb\overline{f}$
1	$\overline{p}bf$, $\overline{p}b\overline{f}$
0	$p\overline{b}f$

$\kappa_{\lvert\Sigma\setminus\{p\}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{b}\overline{f}$
2	-
1	bf, $b\overline{f}$
0	$\overline{b}f$

$\kappa \circ_f^{\Sigma,1} \{p\}\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	bf, $b\overline{f}$
2	-
1	$\overline{b}\overline{f}$
0	$\overline{b}f$

Table 19: Forgetting $p \in \Sigma_{Tweety}$ in the OCF κ over signature Σ_{Tweety}. $\kappa_{\Sigma\setminus\{p\}}$ denotes the marginalization of κ and $\kappa \circ_f^{\Sigma,1} \{p\}$ the forgetting of p by means of $\circ_f^{\Sigma,1}$, respectively. The most plausible interpretations after applying both operators are the same, while the order of the remaining interpretations is inverted.

*results of $\kappa_{\lvert\Sigma\setminus\{p\}}$ and $\kappa \circ_f^{\Sigma,2} \{p\}$ in Tab. 20, and afterwards argue that $\circ_f^{\Sigma,2}$ satisfies **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ even though it does not result in beliefs equivalent to the marginalization.*

*Next, we argue why $\circ_f^{\Sigma,2}$ satisfies **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$. For **(DFPes-1)**$_\Sigma$, we know that it holds due to Prop. 4.7, which concludes*

$$Bel(\kappa) \models Bel(\kappa_{\Sigma\setminus P}) \models Bel(\kappa \circ_f^{\Sigma,2} P).$$

*For **(DFPes-2)**$_\Sigma$, we know that due to the selection function σ the same interpretations are added to the most plausible interpretations of κ and κ', and therefore the assumed subset relation $[\![\kappa]\!] \subseteq [\![\kappa']\!]$ is retained after applying $\circ_f^{\Sigma,2}$ for any $P \subseteq \Sigma$. For **(DFPes-3)**$_\Sigma$, we know that $Bel((\kappa \circ_f^{\Sigma,2} P') \circ_f^{\Sigma,2} P)$ is equivalent to $Bel(\kappa \circ_f^{\Sigma,2} P)$, if $P' \subseteq P$, since the added interpretations $\sigma(P') \cup \sigma(P)$ and $\sigma(P)$ must be equal by definition, and therefore the same interpretations are added to the most plausible interpretations in both cases. Due to $\sigma(P_1 \cup P_2) = \sigma(P_1) \cup \sigma(P_2)$, we can further conclude that **(DFPes-4)**$_\Sigma$ and **(DFPes-5)**$_\Sigma$ do also hold for $\circ_f^{\Sigma,2}$. Finally, we know that **(DFPes-6)**$_\Sigma$ is satisfied as well, since the lifting to the original signature and the intersection with the reduced language cancel each other out, and therefore does*

$\kappa_{\|\Sigma\setminus\{p\}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{b}\overline{f}$
2	-
1	bf, $b\overline{f}$
0	$\overline{b}f$

$\kappa \circ_f^{\Sigma,2} \{p\}\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{b}\overline{f}$
2	-
1	$b\overline{f}$
0	$\overline{b}f$, bf

Table 20: Forgetting $p \in \Sigma_{Tweety}$ in the OCF κ over signature Σ_{Tweety} as given by Tab. 19. $\kappa_{\Sigma\setminus\{p\}}$ denotes the marginalization of κ and $\kappa \circ_f^{\Sigma,2} \{p\}$ the forgetting of p by means of $\circ_f^{\Sigma,2}$. The most plausible interpretations after applying both operators are the same, except for bf, which is added due to the selection function $\sigma(\{p\})$. The remaining interpretations are assigned to the same ranks in both posterior OCFs.

not affect the beliefs of $\kappa \circ_f^{\Sigma,2} P$.

Summary. In summary, we elaborated the relations of the marginalization of OCFs and Delgrande's general forgetting approach [Del17]. By doing so, we proved that the result of forgetting according to Delgrande is equivalent to the beliefs of a marginalization, if the prior beliefs of the OCF are equivalent to the set of formulas we apply Delgrande's forgetting to. Furthermore, we extended the postulates **(DFP-1)**-**(DFP-7)** as originally stated in [Del17] to epistemic states, denoted by **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, and proved that the marginalization satisfies all of them. Thus, the marginalization not only results in equivalent beliefs, but also exactly captures the notions of forgetting according to Delgrande. By means of the shown equivalence, we can further conclude that all relations to other logic-specific forgetting approaches, such as forgetting in propositional logic or literal forgetting as examined by Delgrande in [Del17], can be transferred to the marginalization as well. However, further research on this is needed. Moreover, we illustrated in Ex. 4.1 that the marginalization is not the only operator satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, and at the same time resulting in beliefs equivalent to the result of Delgrande's forgetting approach. Nonetheless, it can be considered the only pure signature reduction operator, since each other operator satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ must induce further changes to the prior OCF, either propositional or conditional. Thus, the induced changes are not restricted to the removal of certain signature elements. Finally, we conclude that Delgrande's general approach is covered by those presented by Kern-Isberner et al. [BKIS+19], and therefore describes one of several kinds of forgetting, rather than providing a comprehensive definition.

4.2 Contraction

In this section, we examine the relations between Delgrande's forgetting approach and c-contractions, which were presented as a kind of forgetting in [BKIS+19] by Kern-Isberner et al. We do so by applying the forgetting postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4) as originally stated by Delgrande [Del17] to c-contractions (Def. 3.36). For this, we first have to generalize the forgetting postulates in a manner that they are applicable to c-contractions. This must be done for the same reasons we had to extend them for the marginalization in Section 4.1 – the postulates describe general ideas, but heavily depend on Delgrande's definition of forgetting. After this, we examine which of the postulates are satisfied by general c-contractions. Due to the almost arbitrary belief changes that can occur when applying c-contractions to epistemic states, we further examine which of the postulates are satisfied by such c-contractions that only induce minimal change to the prior beliefs, which we refer to as minimal change c-contractions (Def. 3.37) in Section 3.2.2. The examinations for the latter are of particular interest, since we already know from [KIBSB17] that they also satisfy the AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40). In case that the properties induced by **(AGMes-1)**-**(AGMes-7)** are sufficient for a certain result, we will not explicitly state it with respect to minimal change c-contractions, but with respect to operators satisfying **(AGMes-1)**-**(AGMes-7)**. Afterwards, we compare the concepts of forgetting and contraction by means of the corresponding postulates. Lastly, after comparing the postulates of both concepts, we focus on the resulting inferences of the approaches by showing that minimal change c-contractions do not results in beliefs equivalent to those of Delgrande's forgetting approach.

4.2.1 Postulates for Forgetting Formulas in Epistemic States

In order to apply the forgetting postulates to c-contractions, we have to generalize them such that they argue about OCFs and their beliefs instead of sets of formulas. Other than the postulates for forgetting signature elements **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ (see Section 4.1 or Appendix A.1), we cannot translate them easily, because c-contractions are belief changes that are not based on a reduction of the corresponding signature. Thus, c-contractions describe a different kind of forgetting, in which we do not want to forget about objects or concepts of our worlds, i.e. signature elements, but instead forget certain information about those objects. Therefore, we cannot make use of the postulates formulated for the marginalization either. Instead, we have to examine the general concepts behind the postulates and express them as the forgetting of formulas in epistemic states. Since this procedure is more general than just re-expressing the postulates to be applicable to c-contractions, we elaborated forgetting postulates for epistemic states **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ that capture the same ideas as **(DFP-1)**-**(DFP-7)** (Th. 3.4) and at the same time are applicable to arbitrary belief change operators. In the following, let Ψ and Φ be epistemic states, $\varphi, \psi \in \mathcal{L}$ formulas, and $\circ_f^{\mathcal{L}}$ an arbitrary belief change operator.

(DFPes-1)$_\mathcal{L}$ $Bel(\Psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$

(DFPes-2)$_\mathcal{L}$ If $Bel(\Psi) \models Bel(\Phi)$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \models Bel(\Phi \circ_f^\mathcal{L} \varphi)$

(DFPes-3)$_\mathcal{L}$ If $\varphi \models \psi$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \equiv Bel((\Psi \circ_f^\mathcal{L} \psi) \circ_f^\mathcal{L} \varphi)$

(DFPes-4)$_\mathcal{L}$ $Bel(\Psi \circ_f^\mathcal{L} (\varphi \vee \psi)) \equiv Bel(\Psi \circ_f^\mathcal{L} \varphi) \cap Bel(\Psi \circ_f^\mathcal{L} \psi)$

(DFPes-5)$_\mathcal{L}$ $Bel(\Psi \circ_f^\mathcal{L} (\varphi \vee \psi)) \equiv Bel((\Psi \circ_f^\mathcal{L} \varphi) \circ_f^\mathcal{L} \psi)$

(DFPes-6)$_\mathcal{L}$ If $\varphi \not\equiv \top$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \not\models \varphi$

In contrast to **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**, a belief change operator $\circ_f^\mathcal{L}$ satisfying **(DFPes-1)$_\mathcal{L}$-(DFPes-6)$_\mathcal{L}$** describes a general forgetting operator that is applied to an epistemic state and a proposition, instead of a subsignature. Since the above-mentioned postulates are frequently used in several sections they can also be found in Appendix A.1. In the following, we explain why **(DFPes-1)$_\mathcal{L}$-(DFPes-6)$_\mathcal{L}$** generalize **(DFP-1)-(DFP-7)** to epistemic states and formulas, while still capturing the same fundamental ideas.

(DFP-1) and **(DFP-2)** can be transferred in a straight-forward manner by substituting the sets of formulas Γ, Γ' with the epistemic states Ψ, Φ, and the deductive closure Cn with the corresponding beliefs. Thus, the resulting postulate **(DFPes-1)$_\mathcal{L}$** is equivalent to the first contraction postulate for epistemic states **(AGMes-1)** (see Section 2.3 or Appendix A.1). Since the beliefs of epistemic states are deductively closed by definition, it is pointless to transfer the idea behind **(DFP-3)** to epistemic states. Notice that due to the omission of **(DFP-3)**, the numbering of the postulates as originally stated by Delgrande is not preserved. Thus, **(DFP-4)-(DFP-6)** correspond to **(DFPes-3)$_\mathcal{L}$-(DFPes-5)$_\mathcal{L}$**. **(DFP-4)-(DFP-6)** cannot be re-expressed in the same straight-forward manner as the first two postulates, which is why we elaborate their main ideas in the following in order to formulate them in the sense of belief changes afterwards.

(DFP-4) states that iteratively forgetting two subsignatures P' and P with $P' \subseteq P$ should result in the same set of formulas as only forgetting P. In our view, the main idea behind this postulate can be described more abstractly as forgetting two pieces of information iteratively, where one piece of information is fully included in the other, results in the same beliefs as just forgetting the more general piece of information. We found this concept to be described most accurately by assuming that a formula ψ is fully included in a formula φ, if φ is more specific than ψ, i.e. $\varphi \models \psi$. Thus, **(DFPes-3)$_\mathcal{L}$** states that consecutively forgetting ψ and φ in Ψ results in beliefs equivalent to those of forgetting φ in Ψ, if we assume $\varphi \models \psi$. Note that this is equivalent to the first postulate for iterated revision **(DP1)** (see Section 2.3 or Appendix A.1) as presented by Darwiche in Pearl [DP97].

(DFP-5) describes that the result of forgetting two subsignatures P and P' at once, i.e. $P \cup P'$, can also be expressed as forgetting P and P' separately and intersecting the results afterwards. Again, we describe the idea of this postulate in a more abstract manner, by saying that forgetting two pieces of information at once yields the same beliefs as forgetting them separately and combining the corresponding results such that the final result contains only those formulas that are included in both of them. The unification of two pieces of information can be expressed as the disjunction of formulas. Even if the conjunction might seem to be the more intuitive choice, it does not capture the idea of unifying two pieces of information. Forgetting the conjunction of two formulas $\varphi \wedge \psi$ would result in a belief set in which $\varphi \wedge \psi$ can no longer be inferred. Nonetheless, either φ or ψ could still be inferred afterwards, since forgetting $\varphi \wedge \psi$ only means, that both formulas cannot be true at the same time. Thus, we combine φ and ψ disjunctively. This way, forgetting results in a belief set in which neither φ nor ψ can be inferred. The final result is then obtained by the conjunction of the two forgetting results. This way the final result contains only those beliefs both interim results agree on. **(DFP-6)** can then be re-expressed following the same ideas as **(DFP-5)**.

Since **(DFP-7)** assumes the forgetting operator to reduce the signature of the prior beliefs to a subsignature, we do not think that it describes any property that is applicable to forgetting propositions, and therefore we omit it. Instead, we like to introduce an additional postulate **(DFPes-6)**$_{\mathcal{L}}$ that describes the success of forgetting a proposition φ, meaning that forgetting should result in an epistemic state that cannot infer φ in case that φ is non-tautologous. We think that it is important to formulate this additional postulate, because it describes the most fundamental idea of forgetting, which was just implicitly given by Delgrande's forgetting postulates. Note that **(DFPes-6)**$_{\mathcal{L}}$ is equivalent to the success postulates for contractions in epistemic states **(AGMes-3)** (see Section 2.3 or Appendix A.1).

After extending **(DFP-1)**-**(DFP-7)** to epistemic states and formulas, we show that the commutativity, associativity and idempotence of Delgrande's forgetting definition (Cor. 3.7,Cor. 3.10) are retained by the generalized postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ (Prop. 4.8). However, notice that these properties are considered with respect to the prior and posterior beliefs instead of the epistemic states themselves.

Proposition 4.8. *Let Ψ be an epistemic state, $\varphi, \psi, \xi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-5)**$_{\mathcal{L}}$, then $\circ_f^{\mathcal{L}}$ satisfies the following properties:*

$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \equiv Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \qquad \text{(Commutativity)}$$
$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi \vee \xi) \equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) \circ_f^{\mathcal{L}} \xi) \qquad \text{(Associativity)}$$
$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \qquad \text{(Idempotence)}$$

Proof of Prop. 4.8.

(Commutativity):

$$\begin{aligned} Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) &\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) && \textbf{(DFPes-5)}_{\mathcal{L}} \\ &\equiv Bel(\Psi \circ_f^{\mathcal{L}} \psi \vee \varphi) && \text{(Commutativity of } \vee) \\ &\equiv Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-5)}_{\mathcal{L}} \end{aligned}$$

(Associativity):

$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi \vee \xi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee (\psi \vee \xi)) \qquad \textbf{(DFPes-5)}_{\mathcal{L}}$$

$$\equiv Bel(\Psi \circ_f^{\mathcal{L}} (\varphi \vee \psi) \vee \xi) \qquad \text{(Associativity of } \vee)$$
$$\equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) \circ_f^{\mathcal{L}} \xi) \qquad \textbf{(DFPes-5)}_{\mathcal{L}}$$

(Idempotence):

$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee \varphi) \qquad \textbf{(DFPes-5)}_{\mathcal{L}}$$
$$\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$$

□

4.2.2 General C-Contractions as Forgetting Operators

In the following, we examine which of the generalized forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ are satisfied by arbitrary c-contractions. Since c-contractions are defined over OCFs, we further refer to the epistemic states Ψ, Ψ' as OCFs κ, κ' over signature Σ. Furthermore, we restrict the examinations to contractions of formulas or conditionals of the form $(\psi|\top)$, respectively, despite the fact that c-contractions are generally defined over conditionals, since the postulates only argue about the forgetting of formulas and the beliefs of epistemic states.

(DFPes-1)$_{\mathcal{L}}$ forms one of the most fundamental ideas of forgetting, saying that knowledge can only be inferred after forgetting, if it could already be inferred by the prior beliefs. It can be shown that even this rather simplistic postulate is not satisfied by arbitrary c-contractions. For a counter example, we consider our epistemic state to be the OCF κ given by Tab. 21 and $\neg p$ as the formula we want to forget.

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\,\overline{f}$
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}\,\overline{b}\,\overline{f}$, $\overline{p}bf, \overline{p}\overline{b}f$

$\kappa_c^{\circ}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	-
3	$p\overline{b}f$
2	$\overline{p}b\overline{f}$
1	$pbf, p\overline{b}\,\overline{f}, \overline{p}\,\overline{b}\,\overline{f}, \overline{p}\overline{b}f, \overline{p}bf$
0	$pb\overline{f}$

Table 21: Left: OCF κ over signature Σ_{Tweety}. Right: Result of forgetting $\neg p$ in κ, $\kappa_c^{\circ} = \kappa \ominus \neg p$, with parameters $\gamma^- = -3$, $\gamma^+ = -1$ and $\kappa_0 = -2$.

By definition of c-contractions, the contraction of a formula is not unique, but can be achieved by multiple c-contractions, which differ in the choice of the parameters γ^-, γ^+ and κ_0 as already explained in Section 3.2.2. The chosen parameters must fulfil the constraints $\gamma^- - \gamma^+ \leq \kappa(\varphi) - \kappa(\neg\varphi)$ and $\kappa_0 = \gamma^- + \kappa(\neg\varphi)$, which correspond to those given in Def. 3.36 when assuming $\varphi \equiv \top$. These constraints only guarantee that forgetting a formula φ results in an OCF that cannot infer φ, which is not sufficient to prevent further belief changes affecting other formulas as well. In our example, the parameters must fulfil the following constraints:

$$\begin{aligned} \kappa_0 &= \gamma^- + \kappa(p), \kappa(\bot) & \gamma^- - \gamma^+ &\leq \kappa(\neg p) - \kappa(p) \\ &= \gamma^- + 1 & &= 0 - 1 \\ & & &= -1 \end{aligned} \tag{4.1}$$

Given these constraints, we can choose the parameters such that the prior beliefs cannot be inferred by the posterior and vice-versa. For this we choose $\gamma^- = -3$, $\gamma^+ = -1$ and $\kappa_0 = -2$ and obtain $\kappa \ominus \neg p$ as given in Tab. 21. The ranks of those interpretations satisfying $\neg p$ are shifted by $-(\gamma^- + 1) + \gamma^+ = 1$, while the interpretations satisfying the contrary are shifted by $-(\gamma^- + 1) + \gamma^- = -1$. This also shows that the impact on the models of p is constant, while the impact on the models of $\neg p$ increases with a greater absolute difference of γ^- and γ^+. When choosing γ^- and γ^+ such that the difference is greater than necessary, we remove the models of $\neg p$ from the most plausible interpretations in κ, which in this case affects all interpretations with rank 0. But in order to satisfy **(DFPes-1)**$_{\mathcal{L}}$, the resulting most plausible interpretations $[\![\kappa_c^\circ]\!]$ should at least consist of those of the prior OCF. Since the contraction of $\neg p$ changed the epistemic state such that none of the previous most plausible interpretations $[\![\kappa]\!]$ are assigned to rank 0 anymore, it is possible to infer formulas from the posterior beliefs that could not be inferred before, e.g. $\kappa_c^\circ \models p \wedge b$, but $\kappa \not\models p \wedge b$. In conclusion, we see that depending on the parameter choice the prior and the posterior beliefs do not necessarily relate to each other. This makes it difficult to argue about the changes induced by the c-contraction aside from the desired behaviour of not being able to infer a certain formula anymore and the principle of conditional preservation, which holds due to the definition of c-changes (Def. 3.35). Further assumptions would be necessary in order to restrict the changes applied to the beliefs corresponding to the OCFs. Darwiche and Pearl already discussed this unpleasant behaviour in the context of iterated belief revision in [DP97], by showing that minimizing conditional changes can result in subsequently performed belief changes, where the second annihilates the effect of the first completely, which may further produce unmotivated propositional changes.

The way the beliefs of an OCF can change due to arbitrary c-contractions is also a problem in **(DFPes-2)**$_{\mathcal{L}}$, even though it assumes the belief set of one of the OCFs to be inferable from the other. There, the assumptions on the prior beliefs cannot guarantee any relation of the resulting beliefs, since the way both belief sets change can be fundamentally different. In general, this is not an undesired behaviour, when applying belief change operators to epistemic states, since the posterior beliefs not only depend on the prior, but also on further properties of the epistemic state. Thus, we can show that **(DFPes-2)**$_{\mathcal{L}}$ is not satisfied by arbitrary c-contractions by giving a counter example. For this, let κ be the OCF given in Tab. 22 and $\kappa' = \kappa$ for reasons of simplicity. We perform two different c-contractions of $\neg p$ in κ. For the first c-contraction, we choose the parameters $\gamma^- = 0$, $\gamma^+ = 1$ and $\kappa_0 = 1$ and

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\overline{f}$
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf, \overline{p}\overline{b}f$

$\kappa_c^\circ(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	-
3	$p\overline{b}f$
2	-
1	$\overline{p}b\overline{f}$, pbf, $p\overline{b}\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf, \overline{p}\overline{b}f$, $pb\overline{f}$

Table 22: Left: OCF κ over signature Σ_{Tweety}. Right: Result of forgetting $\neg p$ in κ, $\kappa_c^\circ = \kappa \ominus \neg p$, with parameters $\gamma^- = 0$, $\gamma^+ = 1$ and $\kappa_0 = 1$.

obtain $\kappa \ominus \neg p$ as given in Tab. 22. Notice that the choice of the parameters for this contraction must satisfy the same constraints as in the given counter example to **(DFPes-1)**$_\mathcal{L}$ (Eq. 4.1), since we are considering the same OCF κ and formula $\neg p$. This contraction results in a belief set that can actually be inferred by the prior beliefs, because the most plausible interpretations in the prior OCF $[\![\kappa]\!]$ are included in $[\![\kappa \ominus \neg p]\!]$. For the second c-contraction $\kappa' \ominus \neg\varphi$, we refer to the contraction described in Tab. 21 above, in which the belief set changed such that its prior and posterior models are disjunct. Comparing the posterior beliefs of the c-contractions described above, we see that they do not preserve the assumed relation of the prior beliefs stated in **(DFPes-2)**$_\mathcal{L}$, because the second contraction replaces $[\![\kappa']\!]$ by $\{pb\overline{f}\}$, whereas the first contraction only extends $[\![\kappa]\!]$ by $\{pb\overline{f}\}$.

$$\gamma^- = 0, \gamma^+ = 1, \kappa_0 = 1: \qquad\qquad \gamma^- = -3, \gamma^+ = -1, \kappa_0 = -2:$$

$$\begin{aligned} & Bel(\kappa \ominus \neg p) \\ \equiv\ & Th([\![\kappa \ominus \neg p]\!]) \\ \equiv\ & Th(\{\overline{p}\overline{b}\overline{f}, \overline{p}bf, \overline{p}\overline{b}f, pb\overline{f}\}) \end{aligned} \qquad\qquad \begin{aligned} & Bel(\kappa' \ominus \neg p) \\ \equiv\ & Th([\![\kappa \ominus \neg p]\!]) \\ \equiv\ & Th(\{pb\overline{f}\}) \end{aligned}$$

Since $[\![\kappa \ominus \neg p]\!]$ is not included in $[\![\kappa' \ominus \neg p]\!]$, we know that $Bel(\kappa' \ominus \neg p)$ cannot be inferred from $Bel(\kappa \ominus \neg p)$ due to Prop. 2.41. Finally, this example illustrates that **(DFPes-2)**$_\mathcal{L}$ is not satisfied by arbitrary c-contractions due to the possible impact on the prior most plausible interpretations.

Next, we show that **(DFPes-3)**$_\mathcal{L}$ is not satisfied by arbitrary c-contractions either, due to the same problem as for both previous forgetting postulates. From the counter example on **(DFPes-1)**$_\mathcal{L}$, we already know that a c-contraction of a formula ψ from an OCF κ can change its most plausible interpretations such that the prior do not relate to the posterior. Furthermore, if we want to contract another formula φ from both κ and $\kappa \ominus \psi$, it is not possible to guarantee any relation between the posterior beliefs, especially not their equivalence, since the contraction is applied to two OCFs with possibly disjunct beliefs, even though we assume $\varphi \models \psi$. In the following, we give a counter example showing that **(DFPes-3)**$_{\mathcal{L}}$ is not satisfied by arbitrary c-contractions. For this, we assume κ as in Tab. 23 and the formulas $\psi \equiv \neg p \vee \neg f$ and $\varphi \equiv \neg p$.

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\overline{f}$
1	$\overline{p}b\overline{f}$, $pb\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$

$\kappa \ominus \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	-
3	$p\overline{b}f$
2	-
1	$\overline{p}b\overline{f}$, pbf, $p\overline{b}\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$, $pb\overline{f}$

$\kappa \ominus \psi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}\overline{f}$
3	$\overline{p}b\overline{f}$, $pb\overline{f}$
2	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$, $p\overline{b}f$
1	-
0	pbf

$\kappa_c^\circ \ominus \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}\overline{f}$
3	$\overline{p}b\overline{f}$, $pb\overline{f}$
2	$\overline{p}\overline{b}\overline{f}$, $\overline{p}bf$, $\overline{p}\overline{b}f$, $p\overline{b}f$
1	-
0	pbf

Table 23: Different contractions of $\psi \equiv \neg p \vee \neg f$ and $\varphi \equiv \neg p$. Top left: OCF κ over signature Σ_{Tweety}. Top right: Result of forgetting $\neg p$ in κ with parameters $\gamma^- = 0$, $\gamma^+ = 1$ and $\kappa_0 = 1$. Bottom left: Result of forgetting $\psi \equiv \neg p \vee \neg f$ in κ with parameters $\gamma^- = -5$, $\gamma^+ = -2$ and $\kappa_0 = -3$. Bottom Right: Result of forgetting $\varphi \equiv \neg p$ in $\kappa_c^\circ = \kappa \ominus \psi$ with parameters $\gamma^- = \gamma^+ = \kappa_0 = 0$.

First, we contract $\psi \equiv \neg p \vee \neg f$ from the initial OCF κ, where the choice of the parameters γ^-, γ^+ and κ_0 must satisfy the following constraints:

$$\kappa_0 = \gamma^- + \kappa(p \wedge f) \qquad \gamma^- - \gamma^+ \leq \kappa(\neg p \vee \neg f) - \kappa(p \wedge f)$$
$$= \gamma^- + 2 \qquad = -2$$

We choose the parameters $\gamma^- = -5$, $\gamma^+ = -2$ and $\kappa_0 = -3$, resulting in $\kappa_c^\circ = \kappa \ominus \neg p \vee \neg f$ (Tab. 23). Since all interpretations in $[\![\kappa]\!]$ satisfy $\neg p \vee \neg f$, their ranks will be increased such that they are no longer part of the most plausible interpretations. On the other hand, the ranks of the models of $p \wedge f$ will be decreased, adding pbf as the only interpretation to rank 0. Therefore, the prior and posterior most plausible interpretations are disjunct. Given the two OCFs κ and κ_c°, we can contract the more specific piece of information $\varphi \equiv \neg p$ from both and result in unrelated posterior beliefs. For the contraction $\kappa \ominus \neg p$ (Tab. 23), we again choose the parameters $\gamma^- = 0$, $\gamma^+ = 1$ and $\kappa_0 = 1$. For the contraction $\kappa_c^\circ \ominus \neg p$ (Tab. 23), we can choose $\gamma^- = \gamma^+ = \kappa_0 = 0$ since $\neg p$ cannot be inferred by κ_c°, and therefore we do not have to change any of the ranks to satisfy the underlying success postulate $\kappa_c^\circ \not\models \neg p$. Since the most plausible interpretations of $\kappa \ominus \varphi$ and $(\kappa \ominus \psi) \ominus \varphi$ are disjunct, their beliefs can neither be equivalent nor be inferred from each other, and therefore **(DFPes-3)**$_\mathcal{L}$ is not satisfied by arbitrary c-contractions.

For **(DFPes-4)**$_\mathcal{L}$, we can give another counter example showing that this postulate cannot be satisfied by arbitrary c-contractions as well. In this case, the problem that occurs is that the contraction of a certain formula can be realized by multiple c-contraction that result in different belief sets. This enables the possibility of choosing the c-contractions of two formulas φ and ψ such that the intersected beliefs of $\kappa \ominus \varphi$ and $\kappa \ominus \psi$ can be different to the beliefs of $\kappa \ominus \varphi \vee \psi$. We illustrate this using the OCFs κ, $\kappa \ominus \neg p$ and $\kappa \ominus \neg p \vee \neg f$ as given in Tab. 23. Since $\neg f$ cannot be inferred by κ, we choose $\gamma^- = \gamma^+ = \kappa_0 = 0$ for the c-contraction of $\neg f$ in κ resulting in $\kappa \ominus \neg f$, which is identical to κ. Comparing the corresponding beliefs

$$\begin{aligned} Bel(\kappa \ominus \neg f) &\equiv Th(\{\overline{p}\overline{b}\overline{f}, \overline{p}bf, \overline{p}\overline{b}f\}),\\ Bel(\kappa \ominus \neg p) &\equiv Th(\{\overline{p}\overline{b}\overline{f}, \overline{p}bf, \overline{p}\overline{b}f, pb\overline{f}\}),\\ Bel(\kappa \ominus \neg f \vee \neg p) &\equiv Th(\{pbf\}), \end{aligned}$$

we can infer $p \wedge b \wedge f$ from $Bel(\kappa \ominus \neg f \vee \neg p)$, but not from $Bel(\kappa \ominus \neg f) \cap Bel(\kappa \ominus \neg p)$. In conclusion, the beliefs of forgetting both formulas at once does not result in equivalent beliefs as the intersection of forgetting both formulas separately. Therefore, we showed that arbitrary c-contractions do not satisfy **(DFPes-4)**$_\mathcal{L}$.

After showing that none of the hitherto postulates are satisfied by arbitrary c-contractions, we show the **(DFPes-5)**$_\mathcal{L}$ is not satisfied either. For this, let κ be as given in Tab. 23 and $\neg f$, $\neg p$ the formulas we want to forget consecutively. For the results of contracting $\neg p$ and $\neg p \vee \neg f$ we also refer to those given in Tab. 23. For the contraction $\kappa \ominus \neg f$, we again choose $\gamma^+ = \gamma^- = \kappa_0 = 0$ as in the counter example on **(DFPes-4)**$_\mathcal{L}$. Therefore, we know that $\kappa \ominus \neg f$ and κ must be identical. Comparing the corresponding beliefs

$$\begin{aligned} &Bel((\kappa \ominus \neg f) \ominus \neg p) \equiv Bel(\kappa \ominus \neg p) \equiv Th(\{\overline{p}\overline{b}\overline{f}, \overline{p}bf, \overline{p}\overline{b}f, pb\overline{f}\}),\\ &Bel(\kappa \ominus \neg f \vee \neg p) \equiv Th(\{pbf\}), \end{aligned}$$

we see that their models are disjunct, and therefore the beliefs of forgetting $\neg f$ and $\neg p$ consecutively and simultaneously are not equivalent. In conclusion, we showed that **(DFPes-5)**$_\mathcal{L}$ is not satisfied by arbitrary c-contractions.

The only forgetting postulate that is actually satisfied by all c-contraction is **(DFPes-6)**$_\mathcal{L}$, since it matches their underlying success postulate, which is guaranteed by the parameter constraints. In conclusion, we showed that none of the forgetting postulates for epistemic states, but **(DFPes-6)**$_\mathcal{L}$, is satisfied by arbitrary c-contractions. We record the results of our examinations for arbitrary c-contractions in Th. 4.9.

Theorem 4.9. *Each propositional c-contraction $\ominus$ satisfies* ***(DFPes-6)$_{\mathcal{L}}$****, but there exist $\ominus$ for which* ***(DFPes-1)$_{\mathcal{L}}$-(DFPes-5)$_{\mathcal{L}}$*** *do not hold.*

The main reason why arbitrary c-contractions are only capable of satisfying **(DFPes-6)$_{\mathcal{L}}$** in general, is that the success postulate of c-contractions is the only constraint the parameters must fulfil. Without further restrictions it is possible to change the beliefs of an epistemic state by contracting a formula in way that the prior and the posterior beliefs do not relate to each other, meaning that the prior beliefs cannot infer the posterior beliefs and vice-versa. In order to tackle this problem, we want to examine the relations between the forgetting postulates and c-contractions that only induce minimal changes to the beliefs.

4.2.3 Minimal Change C-Contractions as Forgetting Operators

As shown in the previous paragraph, arbitrary c-contractions do not satisfy **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**in general, except for **(DFPes-6)$_{\mathcal{L}}$**, due to the changes they are able to induce to the prior beliefs. Therefore, we examine **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$** for such c-contractions that only induce minimal changes to the prior beliefs in the following. For those postulates that cannot be satisfied by minimal change c-contractions either, we determine further restrictions that might be necessary in order to satisfy them, and investigate if weakened variants might be generally satisfied instead. Note that we will focus on minimal change c-contractions (Def. 3.37) in the following examinations, but at the same time will formulate the results as general as possible. Thus, if it sufficient to assume a contraction satisfying **(AGMes-1)**-**(AGMes-7)** (see Section 2.3 or Appendix A.1), we will not explicitly formulate a property for minimal change c-contractions. However, assuming **(AGMes-1)**-**(AGMes-7)** always includes minimal change c-contractions, since we know from Prop. 3.40 that they satisfy these postulates.

Examining (DFPes-1)$_{\mathcal{L}}$ and (DFPes-6)$_{\mathcal{L}}$ for minimal change c-contractions. The first forgetting postulate **(DFPes-1)$_{\mathcal{L}}$** is satisfied by minimal change c-contractions (Lem. 4.10), because it equals the first contraction postulate **(AGMes-1)** for which we already know that it is satisfied by minimal change c-contractions (Prop. 3.40).

Lemma 4.10. *Let $\ominus$ be a minimal change c-contraction, then $\ominus$ satisfies* ***(DFPes-1)$_{\mathcal{L}}$****.*

We refer to [KIBSB17] for a detailed proof and only present the idea behind it at this point. A minimal change c-contraction of a formula φ in an OCF κ only affects the models of $\neg\varphi$ in a way that their ranks are decreased by the minimum absolute value necessary to add them to the most plausible interpretations $[\![\kappa]\!]$. According to Prop. 3.41, we know that the posterior most plausible interpretations $[\![\kappa \ominus \varphi]\!]$ consist of $[\![\kappa]\!]$ unified with $\min\{[\![\varphi]\!], \preceq_\kappa\}$, and therefore $[\![\kappa]\!] \subseteq [\![\kappa \ominus \varphi]\!]$ holds, which is equivalent to **(AGMes-1)**/**(DFPes-1)**$_{\mathcal{L}}$. Furthermore, we know that **(DFPes-6)**$_{\mathcal{L}}$ holds for minimal change c-contraction, since we already showed above that it holds for any arbitrary c-contraction (Th. 4.9).

Examining (DFPes-2)$_{\mathcal{L}}$ for minimal change c-contractions. In the following, we examine **(DFPes-2)**$_{\mathcal{L}}$ for minimal change c-contractions (Def. 3.37). Thereby, we first elaborate why assuming a c-contraction to be a minimal change c-contractions is not sufficient for satisfying **(DFPes-2)**$_{\mathcal{L}}$. Afterwards, we state the importance of the relation between the minimal models in both OCFs, and show that **(DFPes-2)**$_{\mathcal{L}}$ is satisfied, if we assume the minimal models in κ to be included in those in κ' for each formula $\varphi \in \mathcal{L}_\Sigma$.

We start our elaborations on minimal change c-contractions and **(DFPes-2)**$_{\mathcal{L}}$ by showing that they are not capable of satisfying **(DFPes-2)**$_{\mathcal{L}}$ in general. In contrast to arbitrary c-contractions that do not underlie further conditions, minimal change c-contractions allow us to argue about the resulting belief sets $Bel(\kappa \ominus \varphi)$ and $Bel(\kappa' \ominus \varphi)$ as described in **(DFPes-2)**$_{\mathcal{L}}$ more easily, since we know due to Prop. 3.41 how the prior most plausible interpretations are affected by them, i.e. $[\![\kappa \ominus \varphi]\!] = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\}$. However, the expansion of $[\![\kappa]\!]$ by the minimal models that falsify the contracted formula φ is not sufficient to guarantee the fulfilment of the consequence of **(DFPes-2)**$_{\mathcal{L}}$. Its antecedence $Bel(\kappa) \models Bel(\kappa')$ only makes assumptions about the interpretations with rank 0, concretely $[\![\kappa]\!] \subseteq [\![\kappa']\!]$ (Prop. 2.41), and does not argue about the remaining interpretations with a rank greater than 0, since those do not affect the corresponding beliefs. Since there are no assumptions on the interpretations $\omega \in \Omega_\Sigma$ with $\kappa(\omega) > 0$, we cannot guarantee $\min\{[\![\neg\varphi]\!], \preceq_\kappa\}$ to be a subset of $\min\{[\![\neg\varphi]\!], \preceq_{\kappa'}\}$, which would be necessary for the consequence of **(DFPes-2)**$_{\mathcal{L}}$ to hold. Therefore, minimal change c-contractions do not generally satisfy **(DFPes-2)**$_{\mathcal{L}}$ either (Prop. 4.11).

Proposition 4.11. *Let $\ominus$ be a minimal change c-contraction, then there exist OCFs κ, κ' and formulas $\varphi \in \mathcal{L}_\Sigma$, such that*

$$\text{if } Bel(\kappa) \models Bel(\kappa'), \text{ then } Bel(\kappa \ominus \varphi) \models Bel(\kappa' \ominus \varphi) \qquad \textbf{(DFPes-2)}_{\mathcal{L}}$$

does not hold.

In the following, we illustrate that minimal change c-contractions are not capable of generally satisfying **(DFPes-2)**$_{\mathcal{L}}$ (Prop. 4.11) in Ex. 4.2.

Example 4.2. *In this example, we illustrate that it is not sufficient for c-contraction to follow the minimal change paradigm in order satisfy **(DFPes-2)**$_{\mathcal{L}}$. For this, let κ and κ' (Tab. 24) be OCFs over signature $\Sigma_{Tweety} = \{p, b, f\}$ that agree on all rank assignments, except for the rank they assign to the interpretation $p\overline{b}\overline{f}$. κ assigns $p\overline{b}\overline{f}$ to rank 1, while κ' assigns it to rank 2. Due to the equality of their most plausible interpretations $[\![\kappa]\!]$ and $[\![\kappa']\!]$, we know that the beliefs of κ and κ' must be equivalent (Prop. 2.38), and therefore the antecedence of **(DFPes-2)**$_{\mathcal{L}}$, namely $Bel(\kappa) \models Bel(\kappa')$, is satisfied. When we forget $\varphi \equiv \neg p$ in κ and κ', the ranks of the models of p will be decreased such that the minimal models of p are assigned to*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf
1	$pb\overline{f}$, $\overline{p}b\overline{f}$, $p\overline{b}\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}\overline{b}f$, $\overline{p}bf$

$\kappa'(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
5	-
4	$p\overline{b}f$
3	-
2	pbf, $p\overline{b}\overline{f}$
1	$pb\overline{f}$, $\overline{p}b\overline{f}$
0	$\overline{p}\overline{b}\overline{f}$, $\overline{p}\overline{b}f$, $\overline{p}bf$

Table 24: OCFs κ and κ' defined over signature Σ_{Tweety}. κ and κ' agree on all ranks they assign to the interpretations $\omega \in \Omega_{\Sigma_{Tweety}}$, except for $p\overline{b}\overline{f}$.

rank 0 afterwards. This concludes from Prop. 3.41, which states that the posterior most plausible models consist of the prior and the minimal models falsifying $\neg p$:

$$\llbracket \kappa \ominus \neg p \rrbracket = \llbracket \kappa \rrbracket \cup \min\{\llbracket p \rrbracket, \preceq_\kappa\} = \{\overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, \overline{p}bf\} \cup \{pb\overline{f}, p\overline{b}\overline{f}\}$$
$$\llbracket \kappa' \ominus \neg p \rrbracket = \llbracket \kappa' \rrbracket \cup \min\{\llbracket p \rrbracket, \preceq_{\kappa'}\} = \{\overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, \overline{p}bf\} \cup \{pb\overline{f}\}$$

The corresponding minimal models of p are $\{pb\overline{f}, p\overline{b}\overline{f}\}$ for κ, and $\{pb\overline{f}\}$ for κ'. As a consequence, the posterior beliefs after contracting $\neg p$ from κ cannot infer those after the contraction from κ', because there exists an interpretation that is assigned to rank 0 in $\kappa \ominus \neg p$ but not in $\kappa' \ominus \neg p$, which violates the necessary subset relation of their most plausible interpretations, namely $p\overline{b}\overline{f}$.

$$\begin{aligned}
& Bel(\kappa \ominus \neg p) \models Bel(\kappa' \ominus \neg p) & \\
\Leftrightarrow\ & \llbracket \kappa \ominus \neg p \rrbracket \subseteq \llbracket \kappa' \ominus \neg p \rrbracket & \text{(Prop. 2.41)} \\
\Leftrightarrow\ & \{\overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, \overline{p}bf\} \cup \{pb\overline{f}, p\overline{b}\overline{f}\} \subseteq \{\overline{p}\overline{b}\overline{f}, \overline{p}\overline{b}f, \overline{p}bf\} \cup \{pb\overline{f}\} & \lightning
\end{aligned}$$

Ex. 4.2 shows that minimal change c-contractions do not satisfy **(DFPes-2)$_\mathcal{L}$** in general (Prop. 4.11), due to the different minimal models of p that can be added to $\llbracket \kappa \rrbracket$ and $\llbracket \kappa' \rrbracket$, respectively. This means that **(DFPes-2)$_\mathcal{L}$** is satisfied, if the contraction of φ is applied to two OCFs κ and κ' in which the minimal models of $\neg\varphi$ with respect to $\preceq_\kappa$ are included in those with respect to $\preceq_{\kappa'}$. We state this implication in Prop. 4.12.

Proposition 4.12. *Let κ, κ' be OCFs over signature Σ, $\varphi \in \mathcal{L}_\Sigma$ a formula, and $\ominus$ a minimal change c-contraction, then $\ominus$ satisfies*

$$\textit{if } Bel(\kappa) \models Bel(\kappa'), \textit{ then } Bel(\kappa \ominus \varphi) \models Bel(\kappa' \ominus \varphi), \qquad \textbf{(DFPes-2)}_\mathcal{L}$$

if $\min\{\llbracket \neg\varphi \rrbracket, \preceq_\kappa\} \subseteq \min\{\llbracket \neg\varphi \rrbracket, \preceq_{\kappa'}\}$.

Proof of Prop. 4.12. In the following, we show that the consequence of **(DFPes-2)$_{\mathcal{L}}$** $Bel(\kappa \ominus \varphi) \models Bel(\kappa' \ominus \varphi)$ holds under the assumptions stated in Prop. 4.12. We refer to these assumptions as:

$$\begin{aligned} &Bel(\kappa) \models Bel(\kappa') \Leftrightarrow [\![\kappa]\!] \subseteq [\![\kappa']\!] && (\models) \\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_{\kappa'}\} && (\subseteq_{\min}) \end{aligned}$$

$$\begin{aligned} & Bel(\kappa \ominus \varphi) \models Bel(\kappa' \ominus \varphi) && \textbf{(DFPes-2)}_{\mathcal{L}} \\ \Leftrightarrow\ & [\![\kappa \ominus \varphi]\!] \subseteq [\![\kappa' \ominus \varphi]\!] && \text{(Prop. 2.41)} \\ \Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \subseteq [\![\kappa']\!] \cup \min\{[\![\varphi]\!], \preceq_{\kappa'}\} && \text{(Prop. 3.41)} \\ \Leftarrow\ & [\![\kappa]\!] \subseteq [\![\kappa']\!] \text{ and } \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_{\kappa'}\} && \\ \Leftrightarrow\ & (\models) \text{ and } (\subseteq_{\min}) && \end{aligned}$$

Since this holds due to the assumptions $(\models)$ and $(\subseteq_{\min})$, we know that Prop. 4.12 holds. □

Thus, we know that if we want the contraction to generally satisfy **(DFPes-2)$_{\mathcal{L}}$**, the minimal models with respect to $\preceq_\kappa$ must be subsets of those with respect to $\preceq_{\kappa'}$ for every formula $\varphi \in \mathcal{L}_\Sigma$. We formulate this relation in Th. 4.13 below. Since this is only based on the way minimal change c-contractions affect the prior most plausible interpretations, we know that this generally holds for each belief change operator satisfying the contraction postulates **(AGMes-1)**-**(AGMes-7)** (Section 2.3, Appendix A.1).

Theorem 4.13. *Let Ψ, Φ be epistemic states equipped with faithfully assigned total preorders $\preceq_\Psi, \preceq_\Phi$ and $-$ a belief change operator satisfying* ***(AGMes-1)****-****(AGMes-7)****, then the following holds:*

If $\min\{[\![\varphi]\!], \preceq_\Psi\} \subseteq \min\{[\![\varphi]\!], \preceq_\Phi\}$ *for all* $\varphi \in \mathcal{L}_\Sigma$*, then* $-$ *satisfies* ***(DFPes-2)$_{\mathcal{L}}$***

Proof of Th. 4.13. In the following, we refer to the assumptions in Th. 4.13 as

$$\begin{aligned} &Bel(\Psi) \models Bel(\Phi) \Leftrightarrow [\![\Psi]\!] \subseteq [\![\Phi]\!], && (\subseteq_{[\![\cdot]\!]}) \\ &\min\{[\![\varphi]\!], \preceq_\Psi\} \subseteq \min\{[\![\varphi]\!], \preceq_\Phi\} \text{ for all } \varphi \in \mathcal{L}_\Sigma. && (\subseteq_{\min}) \end{aligned}$$

$$\begin{aligned} & Bel(\Psi - \varphi) \models Bel(\Phi - \varphi) && \\ \Leftrightarrow\ & [\![\Psi - \varphi]\!] \subseteq [\![\Phi - \varphi]\!] && \text{(Def. 2.12)} \\ \Leftrightarrow\ & [\![\Psi]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\Psi\} \subseteq [\![\Phi]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\Phi\} && \text{(Th. 3.42)} \\ \Leftarrow\ & [\![\Psi]\!] \subseteq [\![\Phi]\!] \text{ and } \min\{[\![\neg\varphi]\!], \preceq_\Psi\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_\Phi\} && \\ \Leftrightarrow\ & (\subseteq_{[\![\cdot]\!]}) \text{ and } (\subseteq_{\min}) && \end{aligned}$$

Due to $(\subseteq_{\min})$ and $(\subseteq_{[\![\cdot]\!]})$, we know that this subset relation holds. Therefore, $-$ satisfies **(DFPes-2)$_{\mathcal{L}}$**, if the minimal models in Ψ are included in those in Φ for all formulas $\varphi \in \mathcal{L}_\Sigma$. □

In Cor. 4.14, we explicitly state the property from Th. 4.13 for OCFs and minimal change c-contractions, which directly concludes from the fact that minimal change c-contractions satisfy **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40).

Corollary 4.14. *Let κ, κ' be OCFs over signature Σ and $\ominus$ a minimal change c-contraction, then the following holds:*

$$\textit{If } \min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\} \textit{ for all } \varphi \in \mathcal{L}_\Sigma, \textit{ then } \ominus \textit{ satisfies } \textbf{\textit{(DFPes-2)}}_{\mathcal{L}}$$

Notice that the underlying total preorders $\preceq_\kappa$ and $\preceq_{\kappa'}$ do not have to be identical for this. It is sufficient that each relation that holds in κ also holds in κ', i.e. $\preceq_\kappa \subseteq \preceq_{\kappa'}$, which means that it is sufficient for κ to be a refinement of κ' (Def. 2.56). We formalize the relation between the underlying total preorders and the subset relation of the minimal models in Th. 4.15 and prove it in the following.

Theorem 4.15. *Let κ, κ' be OCFs over signature Σ and with corresponding total preorders $\preceq_\kappa, \preceq_{\kappa'}$, then the following holds:*

$$\textit{If } \kappa \sqsubseteq \kappa', \textit{ then } \min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\} \textit{ for all } \varphi \in \mathcal{L}_\Sigma.$$

Proof of Th. 4.15. We prove the implication stated in Th. 4.15 by contraposition. Thereby, we assume that $\min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ does not hold for each $\varphi \in \mathcal{L}_\Sigma$, and show that under this assumption $\kappa \sqsubseteq \kappa'$ cannot hold either. The subset relation $\min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ does not hold for each $\varphi \in \mathcal{L}_\Sigma$, if and only if there exists a formula φ for which there exists an interpretation $\omega \in \Omega_\Sigma$ that is included in the minimal models of φ with respect to $\preceq_\kappa$, but not with respect to $\preceq_{\kappa'}$. Therefore, we distinguish two cases in which the intersection of $\min\{[\![\varphi]\!], \preceq_\kappa\}$ and $\min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ is either empty or not, and show that in both cases κ cannot be a refinement of κ'.

Case $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\} = \emptyset$:
Since the intersection $\min\{[\![\varphi]\!], \preceq_{\kappa'}\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ is empty, we know that there must exist an interpretation $\omega \in \Omega_\Sigma$ with

$$\omega \in \min\{[\![\varphi]\!], \preceq_\kappa\} \text{ and } \omega \notin \min\{[\![\varphi]\!], \preceq_{\kappa'}\}.$$

Furthermore, we can conclude

$$\kappa(\omega) \leq \kappa(\omega'), \text{ for all } \omega' \in [\![\varphi]\!],$$

since ω is a minimal model of φ, and thus there cannot exist a model of φ with a smaller rank than $\kappa(\omega)$. Since ω is not included in $\min\{[\![\varphi]\!], \preceq_{\kappa'}\}$, we know that

$$\text{there exists } \omega' \in [\![\varphi]\!] \text{ with } \kappa'(\omega) \not\leq \kappa'(\omega').$$

Thus, there exists a relation in $\preceq_\kappa$ that does not hold in $\preceq_{\kappa'}$, and therefore κ cannot be a refinement of κ'.

Case $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\} \neq \emptyset$:
In this case we assume that $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ is not empty, and that $\min\{[\![\varphi]\!], \preceq_\kappa\}$ is not a subset of $\min\{[\![\varphi]\!], \preceq_{\kappa'}\}$. Thus, we know again that there exists an interpretation $\omega \in \Omega_\Sigma$ with

$$\omega \in \min\{[\![\varphi]\!], \preceq_\kappa\} \text{ and } \omega \notin \min\{[\![\varphi]\!], \preceq_{\kappa'}\}.$$

Since ω is a minimal model of φ with respect to $\preceq_\kappa$, we know that

$$\kappa(\omega) \leq \kappa(\omega')$$

holds for each $\omega' \in \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$. However, since $\omega \notin \min\{[\![\varphi]\!], \preceq_{\kappa'}$ $\}$, we further know

$$\kappa'(\omega) \not\leq \kappa'(\omega')$$

for each $\omega' \in \min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$. Thus, there exist relations in $\preceq_\kappa$ that do not hold in $\preceq_{\kappa'}$. Therefore, κ cannot be a refinement of κ'.

In conclusion, we showed that if we assume that the minimal models of a formula φ with respect to $\preceq_\kappa$ are not included in those with respect to $\preceq_{\kappa'}$, then $\kappa \sqsubseteq \kappa'$ cannot hold. By means of contraposition, we can further conclude that if $\kappa \sqsubseteq \kappa'$ holds, then $\min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ must hold for every $\varphi \in \mathcal{L}_\Sigma$. □

Moreover, since $\min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\}$ for all $\varphi \in \mathcal{L}_\Sigma$ is sufficient to satisfy **(DFPes-2)**$_\mathcal{L}$ (Th. 4.13,Cor. 4.14), we know in conclusion that **(DFPes-2)**$_\mathcal{L}$ is also satisfied, if κ is a refinement of κ' (Prop. 4.16).

Proposition 4.16. *Let κ, κ' be OCFs over signature Σ, $\varphi \in \mathcal{L}_\Sigma$ a formula and $\ominus$ a minimal change c-contraction, then the following holds:*

*If $\kappa \sqsubseteq \kappa'$, then $\ominus$ satisfies **(DFPes-2)**$_\mathcal{L}$.*

Prop. 4.16 directly concludes from Cor. 4.14 and Th. 4.15, since we know from Th. 4.15 that the refinement relation of two OCFs κ, κ' implies that the minimal models of φ in κ are included in those in κ' for all $\varphi \in \mathcal{L}_\Sigma$, which further implies the fulfilment of **(DFPes-2)**$_\mathcal{L}$ due to Cor. 4.14.

In the following, we want to further elaborate the relations between minimal change c-contractions, minimal models and **(DFPes-2)**$_\mathcal{L}$. For this, we will define a subset relation $\subseteq_{\min,\kappa}$ that relates sets of interpretations to each other according to their minimal elements with respect to $\preceq_\kappa$, and examine some of its order theoretical properties. Finally, we show that we can visualize $\subseteq_{\min,\kappa}$ using Hasse diagrams. This allows us to understand the relations of minimal models in one or even multiple OCFs more easily. We will also illustrate this in several examples and explain how the differences of OCFs reflect in $\subseteq_{\min,\kappa}$.

Apart from the relation stated in Th. 4.15, the underlying total preorders of OCFs do not allow us to directly argue about the relations of minimal models. Knowing that they play an essential role for minimal change c-contractions, and belief changes in general, we define the subset relation of minimal models $\subseteq_{\min,\kappa}$ in Def. 4.17 in order to further examine the connections between minimal change c-contractions, minimal models, and **(DFPes-2)**$_\mathcal{L}$.

Definition 4.17. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$. The* minimal model subset relation $\subseteq_{\min,\kappa}$ *of two sets of interpretations $\Theta, \Theta' \subseteq \Omega_\Sigma$ is defined as*

$$\Theta \subseteq_{\min,\kappa} \Theta' \Leftrightarrow \min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta', \preceq_\kappa\}.$$

Next, we want to examine some of the order theoretical properties of $\subseteq_{\min,\kappa}$. In Th. 4.18 we show that $\subseteq_{\min,\kappa}$ forms a non-total preorder, satisfying reflexivity and transitivity. Thereby, $\subseteq_{\min,\kappa}$ is non-total, because if two sets of interpretations Θ and Θ' are disjunct, their minimal models will be disjunct as well, and thus, neither $\Theta \subseteq_{\min,\kappa} \Theta'$ nor $\Theta' \subseteq_{\min,\kappa} \Theta$ holds. In order to form a partial order, $\subseteq_{\min,\kappa}$ would have to fulfil antisymmetry additionally, which states that if $\Theta \subseteq_{\min,\kappa} \Theta'$ and $\Theta' \subseteq_{\min,\kappa} \Theta$ hold, then Θ and Θ' must be equal. However, the antisymmetry is not satisfied, since $\subseteq_{\min,\kappa}$ is defined over the minimal models of Θ and Θ', and therefore it is possible for the minimal models of Θ and Θ' to be equal, while Θ and Θ' differ.

Theorem 4.18. *The subset relation of minimal models $\subseteq_{\min,\kappa}$ of an OCF κ is a non-total preorder and fulfils reflexivity and transitivity. For all $\Theta, \Theta', \Theta'' \subseteq \Omega$:*

$$\Theta \subseteq_{\min,\kappa} \Theta \qquad \text{(Reflexivity)}$$

$$\textit{If } \Theta \subseteq_{\min,\kappa} \Theta' \textit{ and } \Theta' \subseteq_{\min,\kappa} \Theta'', \textit{ then } \Theta \subseteq_{\min,\kappa} \Theta'' \qquad \text{(Transitivity)}$$

Proof of Th. 4.18. Both the reflexivity and the transitivity of $\subseteq_{\min,\kappa}$ can be traced back to the reflexivity and transitivity of $\subseteq$. By definition of $\subseteq_{\min,\kappa}$, we know that

$$\Theta \subseteq_{\min,\kappa} \Theta \Leftrightarrow \min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta, \preceq_\kappa\}$$

holds for all $\Theta \subseteq \Omega$. Since the subset relation $\subseteq$ is reflexive, we know in conclusion that the subset relation of minimal models $\subseteq_{\min,\kappa}$ must be reflexive as well. The same holds for the transitivity:

$$\begin{aligned} &\text{if } \Theta \subseteq_{\min,\kappa} \Theta' \text{ and } \Theta' \subseteq_{\min,\kappa} \Theta'', \text{ then } \Theta \subseteq_{\min,\kappa} \Theta'' \\ \Leftrightarrow\ &\text{if } \min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta', \preceq_\kappa\} \text{ and } \min\{\Theta', \preceq_\kappa\} \subseteq \min\{\Theta'', \preceq_\kappa\}, \\ &\text{then } \min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta'', \preceq_\kappa\}. \end{aligned}$$

Therefore, $\subseteq_{\min,\kappa}$ fulfils reflexivity and transitivity and forms a non-total preorder. □

Further, we define the unified minimal models of two interpretation sets with respect to $\preceq_\kappa$ (Def. 4.19), allowing us to determine the minimal models when combining two sets of interpretations. Furthermore, the unified minimal models can also be used to constructively determine, whether two interpretations sets Θ, Θ' are in a minimal model subset relation $\Theta \subseteq_{\min,\kappa} \Theta'$. Therefore, we will discuss how the unified minimal models exactly relate to $\subseteq_{\min,\kappa}$ in the following.

Definition 4.19. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$. The* unified minimal models $\Theta \cup_{\min,\kappa} \Theta'$ *of two sets of interpretations $\Theta, \Theta' \subseteq \Omega_\Sigma$ with respect to $\preceq_\kappa$ are defined as*

$$\Theta \cup_{\min,\kappa} \Theta' = \min\{\Theta \cup \Theta', \preceq_\kappa\}.$$

If we look at Θ and Θ' as the models of two formulas φ and ψ, the unified minimal models $\Theta \cup_{\min,\kappa} \Theta'$ describe the minimal models of the disjunction $\varphi \vee \psi$, since $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$ (Lem. 2.10). It is important to note that the unification of Θ and Θ' cannot be performed on the minimal model sets directly. If we would perform the unification on the minimal models of Θ and Θ' instead, the unified minimal models would also contain models of $\varphi \vee \psi$ that are less plausible than some of the other models with respect to $\preceq_\kappa$, and therefore would contain non-minimal models. Thus, the unification must be performed on Θ and Θ' before determining the minimal models. Additionally, we denote the unified minimal models of multiple sets of interpretations $\Theta_0, \ldots, \Theta_n$ with $n \in \mathbb{N}_0$ by

$$\bigcup_{\substack{\min,\kappa \\ \Theta \in \{\Theta_0,\ldots,\Theta_n\}}} \Theta = \Theta_0 \cup_{\min,\kappa} \Theta_1 \cup_{\min,\kappa} \ldots \cup_{\min,\kappa} \Theta_n,$$

analogously to the set unification $\cup$. Given the definition of unified minimal models, we show in Prop. 4.20 that two sets of interpretations Θ, Θ' are in a minimal model subset relation $\Theta \subseteq_{\min,\kappa} \Theta'$, if and only if the unified minimal models in Θ are included in those of Θ'.

Proposition 4.20. *Let κ be an OCF over signature Σ and $\Theta, \Theta' \subseteq \Omega_\Sigma$ sets of interpretations, then the following equivalence holds:*

$$\Theta \subseteq_{\min,\kappa} \Theta' \Leftrightarrow \bigcup_{\substack{\min,\kappa \\ \omega \in \Theta}} \{\omega\} \subseteq \bigcup_{\substack{\min,\kappa \\ \omega' \in \Theta'}} \{\omega'\}$$

Proof of Prop. 4.20.

$$\begin{aligned}
& \bigcup_{\substack{\min,\kappa \\ \omega \in \Theta}} \{\omega\} \subseteq \bigcup_{\substack{\min,\kappa \\ \omega' \in \Theta'}} \{\omega'\} \\
\Leftrightarrow\ & \min\{\bigcup_{\omega \in \Theta} \{\omega\}, \preceq_\kappa\} \subseteq \min\{\bigcup_{\omega' \in \Theta'} \{\omega'\}, \preceq_\kappa\} && \text{(Def. 4.19)} \\
\Leftrightarrow\ & \min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta', \preceq_\kappa\} \\
\Leftrightarrow\ & \Theta \subseteq_{\min,\kappa} \Theta' && \text{(Def. 4.17)}
\end{aligned}$$

□

Prop. 4.20 again states that the minimal model subset relation $\subseteq_{\min,\kappa}$ can be constructively determined by means of the unified minimal models. Starting with those interpretation sets only containing a single interpretation, and then proceeding with the interpretation sets containing two interpretations and so on, until we reach $\Theta = \Omega_\Sigma$. This will also be useful for understanding the later stated examples Ex. 4.3 to 4.6 on how the differences between OCFs reflect in their minimal model subset relations. Furthermore, we know that if both Θ and Θ' are in a minimal model subset relation with their unified minimal models, then the latter are equal to the unification of the minimal models of Θ and Θ' (Prop. 4.21).

Proposition 4.21. *Let κ be an OCF over signature Σ with corresponding total preorder $\preceq_\kappa$ and $\Theta, \Theta' \subseteq \Omega_\Sigma$ sets of interpretations, then the following equivalence holds:*

$$\begin{aligned}&\Theta \cup_{\min,\kappa} \Theta' = \min\{\Theta, \preceq_\kappa\} \cup \min\{\Theta', \preceq_\kappa\}\\ \Leftrightarrow\ &\Theta \subseteq_{\min,\kappa} (\Theta \cup_{\min,\kappa} \Theta') \text{ and } \Theta' \subseteq_{\min,\kappa} (\Theta \cup_{\min,\kappa} \Theta')\end{aligned}$$

Proof of Prop. 4.21.

$$\begin{aligned}
&\Theta \subseteq_{\min,\kappa} (\Theta \cup_{\min,\kappa} \Theta') \text{ and } \Theta' \subseteq_{\min,\kappa} (\Theta \cup_{\min,\kappa} \Theta') \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \subseteq \min\{(\Theta \cup_{\min,\kappa} \Theta'), \preceq_\kappa\} \\
&\text{and } \min\{\Theta', \preceq_\kappa\} \subseteq \min\{(\Theta \cup_{\min,\kappa} \Theta'), \preceq_\kappa\} && \text{(Def. 4.17)} \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \subseteq \min\{\min\{\Theta \cup \Theta', \preceq_\kappa\}, \preceq_\kappa\} \\
&\text{and } \min\{\Theta', \preceq_\kappa\} \subseteq \min\{\min\{\Theta \cup \Theta', \preceq_\kappa\}, \preceq_\kappa\} && \text{(Def. 4.19)} \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta \cup \Theta', \preceq_\kappa\} \\
&\text{and } \min\{\Theta', \preceq_\kappa\} \subseteq \min\{\Theta \cup \Theta', \preceq_\kappa\} \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \cup \min\{\Theta', \preceq_\kappa\} \subseteq \min\{\Theta \cup \Theta', \preceq_\kappa\} \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \cup \min\{\Theta', \preceq_\kappa\} = \min\{\Theta \cup \Theta', \preceq_\kappa\} && \text{(Note)} \\
\Leftrightarrow\ &\min\{\Theta, \preceq_\kappa\} \cup \min\{\Theta', \preceq_\kappa\} = \Theta \cup_{\min,\kappa} \Theta' && \text{(Def. 4.19)}
\end{aligned}$$

(Note): Since we know that both the minimal models of Θ and Θ' are included in the minimal models of $\Theta \cup \Theta'$, there cannot exist another model $\omega \in \Theta \cup \Theta'$ that is minimal in $\Theta \cup \Theta'$, but not in Θ or Θ'. Therefore, the sets must be equal.

□

The property stated in Prop. 4.21 will also be useful for understanding the following examples.

After defining the minimal model subset relation (Def. 4.17) and the unified minimal models of two sets of interpretations (Def. 4.19), we want to illustrate their relations given in Prop. 4.20 and Prop. 4.21. We do so by visualizing $\subseteq_{\min,\kappa}$ as Hasse diagrams for multiple OCFs κ, and examining the relations of their minimal models by means of the unified minimal models. Furthermore, this visualization allows us to understand and argue about the differences of multiple OCFs. This can be of particular interest when examining how an OCF is affected by certain belief changes. Even though this is not part of this work, it shows that using Hasse diagrams for visualising $\subseteq_{\min,\kappa}$ might be potentially interesting in future works. In the following examples Ex. 4.3 to 4.6 we will compare the OCF κ to four other OCFs $\kappa', \kappa'', \kappa''', \kappa''''$, all given in Tab. 25, and thereby elaborate the impact of the OCFs' differences on the corresponding preorders of the minimal models $\subseteq_{\min,\cdot}$. Since Hasse diagrams quickly become very complex with an increasing number of elements, the above-mentioned OCFs only assign ranks to three of the four interpretations possible over the signature $\Sigma = \{a, b\}$. The Hasse diagrams corresponding to the OCFs in

Tab. 25 are illustrated in Figure 3. Before discussing the relations of the different OCFs, we want to note that the property stated in Prop. 4.21 exactly corresponds to the incoming unidirectional edges of a node in Figure 3, which corresponds to a set of interpretations $\Theta \subseteq \Omega_\Sigma$, e.g. $\min\{\{\overline{a}b, a\overline{b}\}, \preceq_{\kappa'''}\} = \min\{\{\overline{a}b\}, \preceq_{\kappa'''}\} \cup \min\{\{a\overline{b}\}, \preceq_{\kappa'''}$ $\}$ in Figure 3c.

Example 4.3. *In this example we consider the OCFs κ and κ' as given in Tab. 25 and the corresponding Hasse diagrams in Figure 3a. First, we take a look at the commonalities and differences between κ and κ'. The beliefs of κ and κ' are equivalent, since their most plausible interpretations both consist of the single interpretation ab:*

$$Bel(\kappa) \equiv Th(\{ab\}) \equiv Bel(\kappa')$$

Comparing the remaining interpretations $a\overline{b}$ and $\overline{a}b$ in κ and κ', we see that their order is inverted, which also affects the order of the minimal models as seen in Figure 3a. The relations of the interpretations that contain ab are not affected by the inversion of the remaining interpretations, because ab is still the most plausible interpretation, and therefore the only minimal model of these sets. The relations that are affected are those between the interpretations that do not contain ab, since the changed order also results in changed minimal models. This concretely affects the interpretation sets $\{\overline{a}b\}, \{a\overline{b}\}$ and $\{\overline{a}b, a\overline{b}\}$. Moreover, the unified minimal models are affected as well. In κ, we see that the unified minimal models of $\{\overline{a}b\}$ and $\{a\overline{b}\}$ are

$$\{\overline{a}b\} \cup_{\min,\kappa} \{a\overline{b}\} = \{a\overline{b}\},$$

due to $a\overline{b} \preceq_\kappa \overline{a}b$, while in κ'

$$\{\overline{a}b\} \cup_{\min,\kappa} \{a\overline{b}\} = \{\overline{a}b\}$$

*holds, due to $\overline{a}b \preceq_{\kappa'} a\overline{b}$. This again illustrates that the assumption $Bel(\kappa) \models Bel(\kappa')$ is not sufficient to satisfy **(DFPes-2)$_\mathcal{L}$**, since forgetting a formula with models $\{\overline{a}b, a\overline{b}\}$ would add different interpretations to rank 0, due to the missing subset relation of the unified minimal models. Concretely, contracting $a \wedge b$ in both κ and κ' would result in*

$$\begin{aligned}
& Bel(\kappa \ominus a \wedge b) \models Bel(\kappa' \ominus a \wedge b) \\
\Leftrightarrow\ & [\![\kappa \ominus a \wedge b]\!] = [\![\kappa' \ominus a \wedge b]\!] && \text{(Prop. 2.41)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg a \vee \neg b]\!], \preceq_\kappa\} = [\![\kappa']\!] \cup \min\{[\![\neg a \vee \neg b]\!], \preceq_{\kappa'}\} && \text{(Prop. 3.41)} \\
\Leftrightarrow\ & \{ab\} \cup \{a\overline{b}\} = \{ab\} \cup \{\overline{a}b\} && \lightning
\end{aligned}$$

In conclusion, we demonstrated in this example that the inversion of the order of all interpretations $\omega \in \Omega_\Sigma$ with $\kappa(\omega) > 0$ affects only the (unified) minimal models of those interpretation sets that do not contain any of the interpretations assigned to rank 0.

Example 4.4. *In this example we consider the OCFs κ and κ'' as given in Tab. 25 and the corresponding Hasse diagrams in Figure 3b. In contrast to κ and κ', for*

$\kappa(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
3	-
2	$\overline{a}b$
1	$a\overline{b}$
0	ab

$\kappa'(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
3	-
2	$a\overline{b}$
1	$\overline{a}b$
0	ab

$\kappa''(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
3	-
2	ab
1	$a\overline{b}$
0	$\overline{a}b$

$\kappa'''(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
2	-
1	$a\overline{b}$, $\overline{a}b$
0	ab

$\kappa''''(\omega)$	$\omega \in \Omega_\Sigma$
∞	-
$\vdots$	-
2	-
1	$a\overline{b}$
0	ab, $\overline{a}b$

Table 25: OCFs over signature $\Sigma = \{a, b\}$, where only the subset $\{ab, a\overline{b}, \overline{a}b\} \subseteq \Omega_\Sigma$ of interpretations is considered.

which only the order of the interpretations $\omega \in \Omega_\Sigma$ with $\kappa(\omega) > 0$ is inverted, κ'' inverts the order of all interpretations ω given by κ. Thus, its beliefs $Bel(\kappa'')$ are neither equivalent to nor can they be inferred by those of κ:

$$Bel(\kappa) \equiv Th(\{ab\}) \not\models Th(\{\overline{a}b\}) \equiv Bel(\kappa'').$$

We see that inverting the order affects the order of the minimal model sets such that none of the relations in $\subseteq_{\min,\kappa}$ hold in $\subseteq_{\min,\kappa''}$.

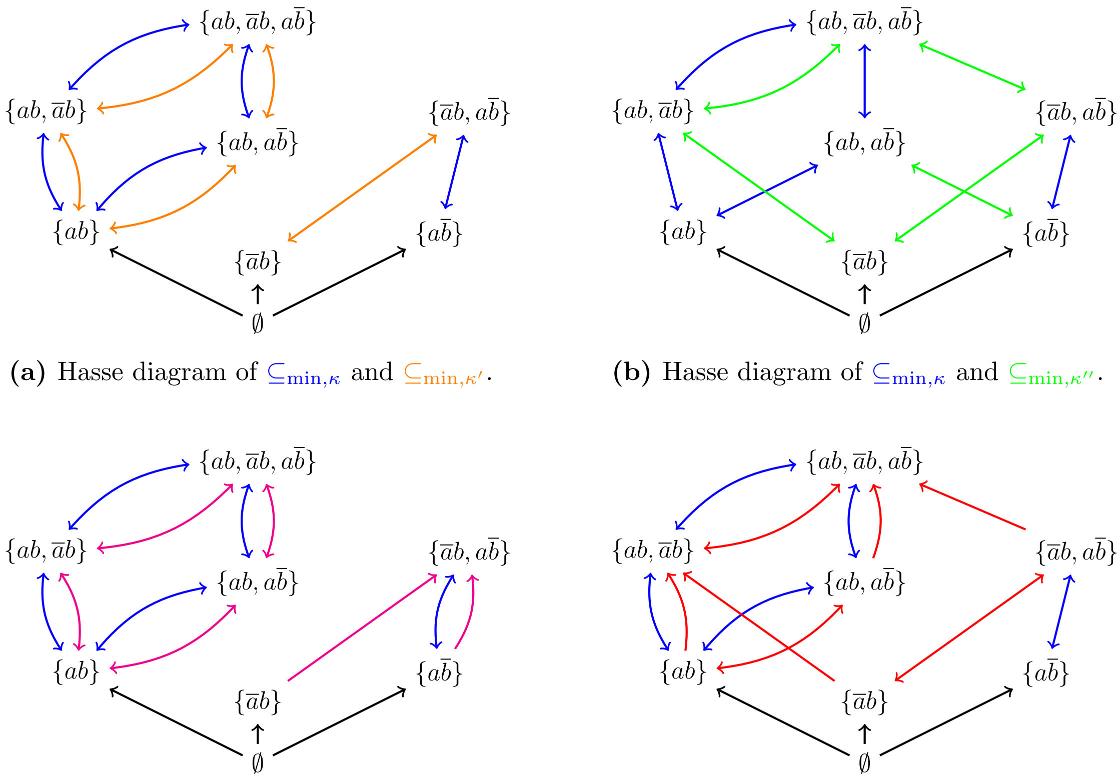
Example 4.5. *In this example we consider the OCFs κ and κ''' as given in Tab. 25 and the corresponding Hasse diagrams in Figure 3c. The OCFs κ and κ''' assign the same ranks to all interpretations, but $\overline{a}b$, for which $\kappa(\overline{a}b) = 2$ and $\kappa'''(\overline{a}b) = 1$ holds. Since the order $\preceq_\kappa$ is preserved by κ''' when changing the rank of $\overline{a}b$, we know that κ is a refinement of κ''' ($\kappa \sqsubseteq \kappa'''$).*

Due to the different ranks of $\overline{a}b$, $\subseteq_{\min,\kappa}$ and $\subseteq_{\min,\kappa'''}$ only differ in the unified minimal models of $\overline{a}b$ and $a\overline{b}$:

$$\{\overline{a}b\} \cup_{\min,\kappa} \{a\overline{b}\} = \{a\overline{b}\},$$
$$\{\overline{a}b\} \cup_{\min,\kappa'''} \{a\overline{b}\} = \{a\overline{b}, \overline{a}b\}.$$

In $\subseteq_{\min,\kappa}$, the unified minimal models of $\overline{a}b$ and $a\overline{b}$ only contain $a\overline{b}$, because $a\overline{b}$ is more plausible than $\overline{a}b$. Since the rank differences of $a\overline{b}$ and $\overline{a}b$ are missing in κ''', which is where κ refines κ''', the unified minimal models of these interpretations equal the unification $\min\{\{a\overline{b}\}, \preceq_{\kappa'''}\} \cup \min\{\{\overline{a}b\}, \preceq_{\kappa'''}\}$. Note that this also corresponds to the property stated in Prop. 4.21, which states that the unified minimal models are equal to the unification of the minimal models, if and only if they are both in a minimal model subset relation with the unified minimal models, i.e.

$$\{\overline{a}b\} \cup_{\min,\kappa'''} \{a\overline{b}\} = \{a\overline{b}, \overline{a}b\}$$



(a) Hasse diagram of $\subseteq_{\min,\kappa}$ and $\subseteq_{\min,\kappa'}$.

(b) Hasse diagram of $\subseteq_{\min,\kappa}$ and $\subseteq_{\min,\kappa''}$.

(c) Hasse diagram of $\subseteq_{\min,\kappa}$ and $\subseteq_{\min,\kappa'''}$.

(d) Hasse diagram of $\subseteq_{\min,\kappa}$ and $\subseteq_{\min,\kappa''''}$.

Figure 3: Hasse diagrams of the minimal model subset relations corresponding to the OCFs in Tab. 25. Each subfigure compares κ to one of the other OCFs. The coloured edges correspond to a certain OCF, whereas the black edges are shared across all OCFs. Bidirectional edges indicate the equality of the corresponding minimal models.

$$\Leftrightarrow \{\bar{a}b\} \subseteq_{\min,\kappa'''} (\{\bar{a}b\} \cup_{\min,\kappa'''} \{a\bar{b}\}) \textit{ and } \{a\bar{b}\} \subseteq_{\min,\kappa'''} (\{\bar{a}b\} \cup_{\min,\kappa'''} \{a\bar{b}\}.$$

*If we now contract $a \wedge b$ in both κ and κ''', which will add the only unified minimal models that are different in both OCFs, namely $\{\bar{a}b\} \cup_{\min,\cdot} \{a\bar{b}\}$, we see that the relation stated by **(DFPes-2)$_{\mathcal{L}}$** holds in this case:*

$$\begin{aligned}
& Bel(\kappa \ominus a \wedge b) \models Bel(\kappa''' \ominus a \wedge b) \\
\Leftrightarrow\ & [\![\kappa \ominus a \wedge b]\!] \subseteq [\![\kappa''' \ominus a \wedge b]\!] && \text{(Prop. 2.41)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg a \vee \neg b]\!], \preceq_\kappa\} \subseteq [\![\kappa''']\!] \cup \min\{[\![\neg a \vee \neg b]\!], \preceq_{\kappa'''}\} && \text{(Prop. 3.41)} \\
\Leftrightarrow\ & \{ab\} \cup \{a\bar{b}\} \subseteq \{ab\} \cup \{a\bar{b}, \bar{a}b\} && \checkmark
\end{aligned}$$

This is due to the fact that κ is a refinement of κ''' (Prop. 4.16). However, it can be seen that this also relates to the relation of the unified minimal models. Each set of unified minimal models in κ is a subset of those in κ'''. In conclusion, we illustrated how the refinement relation of two OCFs affects their minimal model subset relations $\subseteq_{\min,\cdot}$.

Example 4.6. *In this example we consider the OCFs κ and κ'''' as given in Tab. 25 and the corresponding Hasse diagrams in Figure 3d. Just as κ and κ''', κ and κ''''*

only differ in the rank they assign to the interpretation $\overline{a}b$. However, κ is not a refinement of κ'''', since $a\overline{b} \preceq_\kappa \overline{a}b$ holds in κ, while $\overline{a}b \preceq_{\kappa''''} a\overline{b}$ holds in κ''''. Thus, the order of the interpretations in κ is changed by κ''''. Furthermore, since κ'''' assigns $\overline{a}b$ to rank 0, we know that the corresponding beliefs are not equivalent, but $Bel(\kappa) \models Bel(\kappa'''')$ holds instead:

$$Bel(\kappa) \equiv Th(\{ab\}) \models Th(\{ab, \overline{a}b\}) \equiv Bel(\kappa'''')$$

In this case, the unified minimal models of all sets of interpretations in $\subseteq_{\min,\kappa''''}$ that contain ab extend the unified minimal models in $\subseteq_{\min,\kappa}$, because both ab and $\overline{a}b$ are assigned to rank 0 in κ'''', e.g.

$$\{ab\} \cup_{\min,\kappa} \{\overline{a}b\} = \{ab\} \subseteq \{ab, \overline{a}b\} = \{ab\} \cup_{\min,\kappa''''} \{\overline{a}b\},$$
$$\{ab, \overline{a}b\} \cup_{\min,\kappa} \{\overline{a}b, a\overline{b}\} = \{ab\} \subseteq \{ab.\overline{a}b\} = \{ab, \overline{a}b\} \cup_{\min,\kappa''''} \{\overline{a}b, a\overline{b}\}.$$

This can also be seen in Figure 3d, where for example the bidirectional edges between $\{ab\}$ and $\{ab, \overline{a}b\}$ for κ are replaced by two unidirectional incoming edges from $\{ab\}$ and $\{\overline{a}b\}$ to $\{ab, \overline{a}b\}$ for κ''''. However, due to the changed order, this does not hold for all interpretations. The unified minimal models of those interpretations that contain $\overline{a}b$ but not ab do not form extensions of the unified minimal models in $\subseteq_{\min,\kappa}$, e.g.

$$\{\overline{a}b\} \cup_{\min,\kappa} \{a\overline{b}\} = \{a\overline{b}\} \nsubseteq \{\overline{a}b\} = \{\overline{a}b\} \cup_{\min,\kappa''''} \{a\overline{b}\}.$$

In addition to Figure 3, we provide an overview of how the minimal models subset relations and the unified minimal models of the OCFs given in Tab. 25 relate to each other (Tab. 26). This also shows that the subset relation of the unified minimal models of two interpretation sets Θ and Θ' with respect to different OCFs can only hold, if neither Θ nor Θ' contains interpretations for which the subset relation of their minimal models is not satisfied. This connection can be seen in Tab. 26a, where the unified minimal models in κ are not included in those in κ' for any interpretation set that consist of $\overline{a}b$ or $a\overline{b}$, because the minimal models $\{\overline{a}b, a\overline{b}\}$ in κ are not included in the minimal models in κ'.

Examining (DFPes-3)$_\mathcal{L}$ for minimal change c-contractions. Next, we examine **(DFPes-3)$_\mathcal{L}$** for minimal change c-contractions and show that they are not capable of satisfying this postulate in general, but only under further assumptions. Afterwards, we examine the special case in which φ and ψ are equivalent, and show that in this case **(DFPes-3)$_\mathcal{L}$** states the same property as **(AGMes-5)**. While examining **(DFPes-3)$_\mathcal{L}$** for arbitrary c-contraction, we showed that they cannot satisfy this postulate due to the possible changes they induce to the prior beliefs. However, assuming that a c-contraction only induces minimal changes to the prior beliefs is not sufficient for satisfying **(DFPes-3)$_\mathcal{L}$**. When consecutively contracting two formulas φ and ψ with $\varphi \models \psi$, we know that the models of the posterior beliefs $[\![(\kappa \ominus \psi) \ominus \varphi]\!]$ consist of the prior models $[\![\kappa]\!]$ and the minimal models of $\neg\varphi$ and $\neg\psi$ according to Prop. 3.41. Concretely, the posterior models are given by

$$[\![(\kappa \ominus \psi) \ominus \varphi]\!] = [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\varphi]\!], \preceq_{\kappa\ominus\psi}\}.$$

$\Theta \subseteq \Omega$	min	$\cup_{\min,\cdot}$
$\emptyset$	✓	✓
ab	✓	✓
$\overline{a}b$	✓	✗
$a\overline{b}$	✓	✗
$ab, \overline{a}b$	✓	✓
$ab, a\overline{b}$	✓	✓
$\overline{a}b, a\overline{b}$	✗	✗
$ab, \overline{a}b, a\overline{b}$	✓	✓

(a) Comparison of κ and κ'.

$\Theta \subseteq \Omega$	min	$\cup_{\min,\cdot}$
$\emptyset$	✓	✓
ab	✓	✗
$\overline{a}b$	✓	✗
$a\overline{b}$	✓	✗
$ab, \overline{a}b$	✗	✗
$ab, a\overline{b}$	✗	✗
$\overline{a}b, a\overline{b}$	✗	✗
$ab, \overline{a}b, a\overline{b}$	✗	✗

(b) Comparison of κ and κ''.

$\Theta \subseteq \Omega$	min	$\cup_{\min,\cdot}$
$\emptyset$	✓	✓
ab	✓	✓
$\overline{a}b$	✓	✓
$a\overline{b}$	✓	✓
$ab, \overline{a}b$	✓	✓
$ab, a\overline{b}$	✓	✓
$\overline{a}b, a\overline{b}$	✓	✓
$ab, \overline{a}b, a\overline{b}$	✓	✓

(c) Comparison of κ and κ'''.

$\Theta \subseteq \Omega$	min	$\cup_{\min,\cdot}$
$\emptyset$	✓	✓
ab	✓	✓
$\overline{a}b$	✓	✗
$a\overline{b}$	✓	✗
$ab, \overline{a}b$	✓	✓
$ab, a\overline{b}$	✓	✓
$\overline{a}b, a\overline{b}$	✗	✗
$ab, \overline{a}b, a\overline{b}$	✓	✓

(d) Comparison of κ and κ''''.

Table 26: Relations between the minimal model subset relation and the unified minimal models of the OCFs illustrated in Tab. 25 and Figure 3. The min column indicates whether the minimal models of Θ in κ are a subset of those in the other OCF, i.e. $\min\{\Theta, \preceq_\kappa\} \subseteq \min\{\Theta, \preceq_\cdot\}$. The $\cup_{\min,\cdot}$ column indicates whether the unified minimal models of Θ and all other subsets Θ' in κ are subsets of the unified minimal models in the other OCF, i.e. $\Theta \cup_{\min,\kappa} \Theta' \subseteq \Theta \cup_{\min,\cdot} \Theta'$.

Notice that the minimal models of $\neg\varphi$ are determined with respect to $\preceq_{\kappa\ominus\psi}$, since we have forgotten ψ before. Since **(DFPes-3)**$_\mathcal{L}$ assumes $\varphi \models \psi$, we know that $[\![\varphi]\!] \subseteq [\![\psi]\!]$ holds (Def. 2.12), which is equivalent to $[\![\neg\psi]\!] \subseteq [\![\neg\varphi]\!]$ due to Lem. 2.15. Therefore, we know that contracting ψ from κ in a minimal way adds some of the models of $\neg\varphi$ to rank 0. Note that these models do not necessarily have to be minimal models of $\neg\varphi$, what would be the case if $\kappa(\neg\varphi) < \kappa(\neg\psi)$. Contracting φ afterwards will not affect the OCF $\kappa \ominus \psi$, because there already exist models of $\neg\varphi$ with rank 0 due to $[\![\neg\psi]\!] \subseteq [\![\neg\varphi]\!]$. Since it cannot be guaranteed that the contraction of ψ adds all minimal models of $\neg\varphi$ to $[\![\kappa]\!]$, we know that the minimal models added to the prior most plausible interpretations due to the minimal change c-contractions are not equal. Thus, the posterior beliefs cannot be equivalent. Furthermore, we can especially conclude that neither of the posterior beliefs can be inferred from each other, since it is also possible that the minimal models of $\neg\varphi$ and $\neg\psi$ are disjunct, even though $[\![\neg\psi]\!] \subseteq [\![\neg\varphi]\!]$ holds. This shows that the antecedence stated in **(DFPes-3)**$_{\mathcal{L}}$, namely $\varphi \models \psi$, is not sufficient for minimal change c-contraction to satisfy the corresponding conclusion. In Prop. 4.22, we state that a minimal change c-contraction only satisfies **(DFPes-3)**$_{\mathcal{L}}$, if at least one of the there stated three conditions is satisfied. We will discuss these conditions in the following.

Proposition 4.22. *Let $\ominus$ be a minimal change c-contraction, then the following holds:*

> *If $\ominus$ satisfies **(DFPes-3)**$_{\mathcal{L}}$,*
> *then* $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ *or* $\kappa(\neg\varphi) = 0 = \kappa(\neg\psi)$ *or* $\varphi \not\models \psi$

Proof of Prop. 4.22. We prove Prop. 4.22 by means of contrapositions. Thus, we show that a minimal change c-contraction $\ominus$ does not satisfy **(DFPes-3)**$_{\mathcal{L}}$, if the minimal models of $\neg\varphi$ and $\neg\psi$ are not equal, at least $\neg\varphi$ or $\neg\psi$ is assigned to a rank greater than 0, and $\varphi \models \psi$. Formally, this can be expressed as

$$\begin{aligned}
&\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ and } (\kappa(\neg\varphi) > 0 \text{ or } \kappa(\neg\psi) > 0) \text{ and } \varphi \models \psi \\
&\Rightarrow \neg(\varphi \models \psi \Rightarrow Bel(\kappa \ominus \varphi) \equiv Bel((\kappa \ominus \psi) \ominus \varphi)),
\end{aligned}$$

and is equivalent to

$$\begin{aligned}
&\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ and } (\kappa(\neg\varphi) > 0 \text{ or } \kappa(\neg\psi) > 0) \text{ and } \varphi \models \psi \\
&\Rightarrow \neg(\varphi \models \psi \Rightarrow Bel(\kappa \ominus \varphi) \equiv Bel((\kappa \ominus \psi) \ominus \varphi)) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ and } (\kappa(\neg\varphi) > 0 \text{ or } \kappa(\neg\psi) > 0) \text{ and } \varphi \models \psi \\
&\Rightarrow \neg(\varphi \not\models \psi \text{ or } Bel(\kappa \ominus \varphi) \equiv Bel((\kappa \ominus \psi) \ominus \varphi)) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ and } (\kappa(\neg\varphi) > 0 \text{ or } \kappa(\neg\psi) > 0) \text{ and } \varphi \models \psi \\
&\Rightarrow (\varphi \models \psi \text{ and } Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi)) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ or } \kappa(\neg\varphi) = 0 = \kappa(\neg\psi) \text{ or } \varphi \not\models \psi \\
&\text{or } (\varphi \models \psi \text{ and } Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi)) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ or } \kappa(\neg\varphi) = 0 = \kappa(\neg\psi) \\
&\text{or } ((\varphi \not\models \psi \text{ or } \varphi \models \psi) \text{ and } (\varphi \not\models \psi \text{ or } Bel(\kappa \ominus \varphi)) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi)) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ or } \kappa(\neg\varphi) = 0 = \kappa(\neg\psi) \\
&\text{or } \varphi \not\models \psi \text{ or } Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi) \\
\Leftrightarrow\ &\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} \text{ and } (\kappa(\neg\varphi) > 0 \text{ or } \kappa(\neg\psi) > 0) \text{ and } \varphi \models \psi \\
&\Rightarrow Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi).
\end{aligned} \tag{4.2}$$

For the stated antecedence, there are two cases to distinguish, since $\kappa(\neg\varphi) > 0$ or $\kappa(\neg\psi) > 0$ is true, if $\kappa(\neg\psi) = 0 = \kappa(\neg\varphi)$ or $\kappa(\neg\psi) > 0 = \kappa(\neg\varphi)$ holds. Technically, $\kappa(\neg\varphi) > 0$ or $\kappa(\neg\psi) > 0$ would also true if $\kappa(\neg\varphi) > 0 = \kappa(\neg\psi)$ holds. However, this case is excluded by the assumption $\varphi \models \psi$, which is equivalent to $\neg\psi \models \neg\varphi$ due to Lem. 2.10, since this means that each model of $\neg\psi$ is also a model of $\neg\varphi$. Thus, if $\kappa(\neg\psi) = 0$, then $\kappa(\neg\varphi) = 0$ must hold as well. In the further course, we show that the implication in Eq. 4.2 holds in both of these cases.

Case $\kappa(\neg\psi) > 0, \kappa(\neg\varphi) = 0$:

$$\begin{aligned}
& Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi) & \\
\Leftrightarrow\ & [\![\kappa \ominus \varphi]\!] \neq [\![(\kappa \ominus \psi) \ominus \varphi]\!] & \text{(Prop. 2.38)} \\
\Leftrightarrow\ & [\![\kappa]\!] \neq [\![(\kappa \ominus \psi) \ominus \varphi]\!] & \text{(Lem. 3.38)} \\
\Leftrightarrow\ & [\![\kappa]\!] \neq [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\varphi]\!], \preceq_{\kappa\ominus\psi}\} & \text{(Prop. 3.41)} \\
\Leftarrow\ & [\![\kappa]\!] \neq [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} & \\
\Leftarrow\ & [\![\kappa]\!] \subset [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} & \\
\Leftarrow\ & [\![\kappa]\!] \subset [\![\kappa \ominus \psi]\!] & \text{(Prop. 3.41)}
\end{aligned}$$

Since $\kappa(\neg\psi) > 0$, we know that there do not exist models of $\neg\psi$ that are assigned to rank 0, and therefore $[\![\kappa]\!] \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} = \emptyset$. Thus, $[\![\kappa]\!]$ is a subset of, but not equal to $[\![\kappa \ominus \psi]\!]$, and therefore we showed that in this case to a minimal change c-contraction cannot satisfy **(DFPes-3)**$_\mathcal{L}$.

Case $\kappa(\neg\psi) > 0, \kappa(\neg\varphi) > 0$:

$$\begin{aligned}
& Bel(\kappa \ominus \varphi) \not\equiv Bel((\kappa \ominus \psi) \ominus \varphi) & \\
\Leftrightarrow\ & [\![\kappa \ominus \varphi]\!] \neq [\![(\kappa \ominus \psi) \ominus \varphi]\!] & \text{(Prop. 2.38)}
\end{aligned}$$

By means of the contraction $\kappa \ominus \psi$ the minimal models of ψ with respeect to $\preceq_\kappa$ will be added to the most plausible interpretations (Prop. 3.41), and thus $\kappa \ominus \psi\ (\neg\psi) = 0$. As already mentioned above, we know that $[\![\neg\psi]\!] \subseteq [\![\neg\varphi]\!]$ holds due to the assumption $\varphi \models \psi$. Therefore, $\kappa \ominus \psi\ (\neg\varphi) = 0$ holds as well, which concludes that the subsequently performed contraction of φ does not affect $\kappa \ominus \psi$ (Lem. 3.38):

$$\begin{aligned}
& [\![\kappa \ominus \varphi]\!] \neq [\![(\kappa \ominus \psi) \ominus \varphi]\!] & \text{(Prop. 2.38)} \\
\Leftrightarrow\ & [\![\kappa \ominus \varphi]\!] \neq [\![\kappa \ominus \psi]\!] & \text{(Lem. 3.38)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} & \text{(Prop. 3.41)} \\
\Leftrightarrow\ & \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\} & (\kappa(\neg\psi), \kappa(\neg\varphi) > 0,\ \text{Prop. 2.37})
\end{aligned}$$

Since $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ holds by assumption, we also showed that in the second case a minimal change c-contraction cannot satisfy **(DFPes-3)**$_\mathcal{L}$.

In conclusion, we showed that a minimal change c-contraction cannot satisfy **(DFPes-3)**$_\mathcal{L}$, if we assume $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\psi]\!], \preceq_\kappa\}$, $(\kappa(\neg\varphi) > 0$ or $\kappa(\neg\psi) > 0)$ and $\varphi \models \psi$. By means of contraposition, this means that a minimal change c-contraction only satisfies **(DFPes-3)**$_\mathcal{L}$, if $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa$ $\}$, $\kappa(\neg\varphi) = 0 = \kappa(\neg\psi)$ or $\varphi \not\models \psi$ holds. □

If a minimal change c-contraction satisfies **(DFPes-3)**$_\mathcal{L}$, we know that this can only be the case, if at least one of the following conditions holds. The first condition is the falsification of the antecedence of **(DFPes-3)**$_\mathcal{L}$, since the there stated implication is always true if the antecedence is false. In case that the antecedence $\varphi \models \psi$ holds, we know that one of the other two conditions must hold, i.e. $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ or $\kappa(\neg\varphi) = 0 = \kappa(\neg\psi)$. The equality of the minimal models $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ states that the models added to the prior most plausible interpretations are exactly the same, and thus the posterior beliefs must be equivalent. However, this relation of the minimal models cannot be guaranteed by $\varphi \models \psi$. Lastly, if the antecedence $\varphi \models \psi$ holds, but the minimal models of $\neg\varphi$ and $\neg\psi$ are not equal, then we can conclude that neither φ nor ψ was believed by the prior OCF κ, and therefore none of the performed contractions affect κ.

After showing that **(DFPes-3)$_\mathcal{L}$** only holds for minimal change c-contractions under further assumptions, we want to examine the special case in which $\varphi \equiv \psi$. When first contracting ψ in an OCF κ and φ afterwards with respect to the minimal change paradigm, we know that the second contraction will not affect $\kappa \ominus \psi$, since models of $\neg\varphi$ are already assigned to rank 0 due to $[\![\neg\psi]\!] \subseteq [\![\neg\varphi]\!]$, meaning that $\kappa\ominus\psi$ does not believe φ in the first place, and therefore it does not have to be changed. Given the special case $\varphi \equiv \psi$, we know that the contraction of ψ adds all minimal models of $\neg\varphi$ to $[\![\kappa]\!]$ since the equivalence of φ and ψ implies $[\![\neg\psi]\!] = [\![\neg\varphi]\!]$. In conclusion, we know that forgetting two equivalent formulas also results in equivalent posterior beliefs. Thus, when assuming $\varphi \equiv \psi$, we know that **(DFPes-3)$_\mathcal{L}$** and **(AGMes-5)** (see Section 2.3 or Appendix A.1) are equivalent (Lem. 4.23).

Lemma 4.23. *Let $\varphi, \psi \in \mathcal{L}$ be formulas. If $\varphi \equiv \psi$ holds, then **(DFPes-3)$_\mathcal{L}$** is equivalent **(AGMes-5)**.*

Since we know from Prop. 3.40 that minimal change c-contractions satisfy **(AGMes-5)**, we especially know that under the assumption $\varphi \equiv \psi$, minimal change c-contractions satisfy **(DFPes-3)$_\mathcal{L}$** as well (Lem. 4.24).

Lemma 4.24. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\ominus$ a minimal change c-contraction, then $\ominus$ satisfies **(DFPes-3)$_\mathcal{L}$**, if $\varphi \equiv \psi$.*

Examining (DFPes-4)$_\mathcal{L}$ for minimal change c-contractions. For **(DFPes-4)$_\mathcal{L}$** we show that minimal change c-contractions do not satisfy **(DFPes-4)$_\mathcal{L}$** in general, but only under further assumptions on the contracted formulas. Thus, the assumption of a minimal change c-contraction is not sufficient to satisfy **(DFPes-4)$_\mathcal{L}$**. Since the equivalence stated in **(DFPes-4)$_\mathcal{L}$** can be traced back to the equality of the posterior models, i.e.

$$[\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} = ([\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\}) \cup ([\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\}),$$

we show in Prop. 4.25 below, that minimal change c-contractions only satisfy **(DFPes-4)$_\mathcal{L}$** in case that at least $\neg\psi$ or $\neg\varphi$ is assigned to rank 0 or in case that both are assigned to the same rank. In the following, we will discuss these conditions more detailed.

Proposition 4.25. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\ominus$ a minimal change c-contraction, then $\ominus$ satisfies*

$$Bel(\kappa \ominus \varphi \vee \psi) \equiv Bel(\kappa \ominus \varphi) \cap Bel(\kappa \ominus \psi), \qquad \textbf{(DFPes-4)}_\mathcal{L}$$

only if $\kappa(\neg\varphi) = \kappa(\neg\psi)$ or $\kappa(\neg\varphi) = 0$ or $\kappa(\neg\psi) = 0$.

Proof of Prop. 4.25. First of all, we state that $\mathbf{(DFPes\text{-}4)}_{\mathcal{L}}$ holds if and only if the prior most plausible interpretations $[\![\kappa]\!]$ together with the minimal models of $\neg\varphi \wedge \neg\psi$ are equal to $[\![\kappa]\!]$ together with the minimal models of $\neg\varphi$ and $\neg\psi$:

$$
\begin{aligned}
& Bel(\kappa \ominus \varphi \vee \psi) \equiv Bel(\kappa \ominus \varphi) \cap Bel(\kappa \ominus \psi) && \mathbf{(DFPes\text{-}4)}_{\mathcal{L}} \\
\Leftrightarrow\; & Th([\![\kappa \ominus \varphi \vee \psi]\!]) \equiv Th([\![\kappa \ominus \varphi]\!]) \cap Th([\![\kappa \ominus \psi]\!]) && \text{(Prop. 2.38)} \\
\Leftrightarrow\; & Th([\![\kappa \ominus \varphi \vee \psi]\!]) \equiv Th([\![\kappa \ominus \varphi]\!] \cup [\![\kappa \ominus \psi]\!]) && \text{(Lem. 2.25)} \\
\Leftrightarrow\; & [\![\kappa \ominus \varphi \vee \psi]\!] = [\![\kappa \ominus \varphi]\!] \cup [\![\kappa \ominus \psi]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\; & [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \\
& = ([\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\}) \cup ([\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\}) && \text{(Prop. 2.38)} \\
\Leftrightarrow\; & [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \\
& = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\}
\end{aligned}
$$

In the following, we prove Prop. 4.25 by means of a contraposition. Thus, we show

if $\kappa(\neg\varphi) \neq \kappa(\neg\psi)$ and $\kappa(\neg\varphi) > 0$ and $\kappa(\neg\psi) > 0$,
then $[\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \neq [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\}$.

We refer to $\kappa(\neg\varphi) \neq \kappa(\neg\psi)$ as $(\kappa_{\neq})$ and $\kappa(\neg\varphi) > 0$ and $\kappa(\neg\psi) > 0$ as $(\kappa_{>0})$. Due to $(\kappa_{>0})$, we know that the minimal models added to $[\![\kappa]\!]$ are not included in $[\![\kappa]\!]$, since this would require $\neg\psi$ and $\neg\varphi$ to be assigned to rank 0.

$$
\begin{aligned}
& [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \neq [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} \\
\Leftrightarrow\; & \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} && (\kappa_{>0})
\end{aligned}
$$

From Lem. 2.51 together with the assumption $(\kappa_{\neq})$, we know that the intersection of the minimal models $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ must be empty $(= \emptyset)$. If we then assume $\kappa(\neg\varphi) < \kappa(\neg\psi)$ without loss of generality, we can further conclude $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap [\![\neg\psi]\!] = \emptyset$, because otherwise $\neg\psi$ could not be assigned to a rank greater than that of $\neg\varphi$. Due to this, we especially know $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} = \emptyset$, since $\min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \subseteq [\![\neg\psi]\!]$. Therefore, we know that $\min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \neq \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ holds. In conclusion, we showed that $\mathbf{(DFPes\text{-}4)}_{\mathcal{L}}$ cannot hold, if we assume $(\kappa_{>0})$ and $(\kappa_{\neq})$. Thus, by means of contraposition, we know that Prop. 4.25 holds. □

The equivalence stated in $\mathbf{(DFPes\text{-}4)}_{\mathcal{L}}$ depends on the minimal models of $\neg\varphi$, $\neg\psi$ and $\neg\varphi\wedge\neg\psi$ added to the prior most plausible interpretations due to the contraction. The problem that occurs at this point is that the minimal models of $\neg\varphi \wedge \neg\psi$ do not necessarily have to relate to those of $\neg\varphi$ and $\neg\psi$. Thus, the minimal models of $\neg\varphi \wedge \neg\psi$ and those of $\neg\varphi$ and $\neg\psi$ are potentially disjunct. This concludes that if $\mathbf{(DFPes\text{-}4)}_{\mathcal{L}}$ holds for a minimal change c-contraction, then we know that both none of the contractions affect the prior OCF κ, since both φ and ψ are not believed in the first place, or the minimal models of $\neg\varphi \wedge \neg\psi$ must be equal to the unification of those of $\neg\varphi$ and $\neg\psi$. The latter can be guaranteed when assuming $\kappa(\neg\varphi) = \kappa(\neg\psi)$. We want to illustrate the relation between the minimal models of $\neg\varphi$, $\neg\psi$ and $\neg\varphi \wedge \neg\psi$ in Ex. 4.7.

Example 4.7. *In the following, we illustrate how the fulfilment of* ***(DFPes-4)$_\mathcal{L}$*** *relates to the minimal models of $\neg\varphi$, $\neg\psi$ and $\neg\varphi \wedge \neg\psi$. For this, we consider the prior OCF κ as given in Tab. 27 below, and the formulas $\varphi \equiv f, \psi \equiv p$.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	$p\overline{b}\overline{f}$, $\overline{p}b\overline{f}$, $pb\overline{f}$, $\overline{p}\overline{b}\overline{f}$
1	$\overline{p}bf$, $\overline{p}\overline{b}f$
0	pbf, $p\overline{b}f$

Table 27: OCF κ over signature Σ_{Tweety}.

Note that κ neither assigns $\neg p$ nor $\neg f$ to rank 0. Thus, we know according to Prop. 4.25 that if a minimal change c-contraction satisfies ***(DFPes-4)$_\mathcal{L}$****, then $\kappa(\neg f) = \kappa(\neg p)$ must hold. Next, we want to illustrate that* ***(DFPes-4)$_\mathcal{L}$*** *cannot be satisfied, since $\kappa(\neg f) = 2 \neq 1 = \kappa(\neg p)$ holds.*

As already mentioned above, we know that the equivalence stated in ***(DFPes-4)$_\mathcal{L}$*** *holds, if and only if the posterior most plausible interpretations of the minimal change c-contractions are equal, i.e.*

$$[\![\kappa]\!] \cup \min\{[\![\neg f \wedge \neg p]\!], \preceq_\kappa\} = ([\![\kappa]\!] \cup \min\{[\![\neg f]\!], \preceq_\kappa\}) \cup ([\![\kappa]\!] \cup \min\{[\![\neg p]\!], \preceq_\kappa\}),$$

which is furthermore equivalent to

$$\min\{[\![\neg f \wedge \neg p]\!], \preceq_\kappa\} = \min\{[\![\neg f]\!], \preceq_\kappa\} \cup \min\{[\![\neg p]\!], \preceq_\kappa\}, \tag{4.3}$$

since $[\![\kappa]\!]$ does not consist of models of $\neg f$ or $\neg p$. For the minimal models stated above, we know that they are equal to

$$\begin{aligned}\min\{[\![\neg f \wedge \neg p]\!], \preceq_\kappa\} &= \{\overline{p}b\overline{f}, \overline{p}\overline{b}\overline{f}\},\\ \min\{[\![\neg f]\!], \preceq_\kappa\} &= \{p\overline{b}\overline{f}, \overline{p}b\overline{f}, pb\overline{f}, \overline{p}\overline{b}\overline{f}\},\\ \min\{[\![\neg p]\!], \preceq_\kappa\} &= \{\overline{p}bf, \overline{p}\overline{b}f\}.\end{aligned}$$

Thus, Eq. 4.3 does not hold, since

$$\{\overline{p}b\overline{f}, \overline{p}\overline{b}\overline{f}\} \neq \{p\overline{b}\overline{f}, \overline{p}b\overline{f}, pb\overline{f}, \overline{p}\overline{b}\overline{f}\} \cup \{\overline{p}bf, \overline{p}\overline{b}f\}.$$

This illustrates, that ***(DFPes-4)$_\mathcal{L}$*** *cannot be satisfied if $\neg p$ and $\neg f$ are assigned to different ranks greater than 0, since this implies that the minimal models of $\neg f \wedge \neg p$ and those of $\neg f$ and $\neg p$ are disjunct.*

Examining (DFPes-5)$_{\mathcal{L}}$ for minimal change c-contractions. Next, we examine **(DFPes-5)$_{\mathcal{L}}$** for minimal change c-contraction. For this, we show that **(DFPes-5)$_{\mathcal{L}}$** is satisfied under further assumptions on the minimal models of the formulas we would like to forget (Th. 4.26). Furthermore, we give an example why minimal change c-contractions do not satisfy **(DFPes-5)$_{\mathcal{L}}$** in general, if those further assumptions do not hold.

Theorem 4.26. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas and $\ominus$ a minimal change c-contraction, then $\ominus$ satisfies*

$$Bel(\kappa \ominus \varphi \vee \psi) \equiv Bel((\kappa \ominus \varphi) \ominus \psi), \qquad (\textbf{(DFPes-5)}_{\mathcal{L}})$$

if $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} \neq \emptyset$.

Proof of Th. 4.26. In the following, we distinguish two cases. In the first case we assume $\kappa(\neg\varphi \wedge \neg\psi) = 0$, whereas we assume $\kappa(\neg\varphi \wedge \neg\psi) > 0$ in the second case.

$$\begin{aligned}
& Bel(\kappa \ominus \varphi \vee \psi) \equiv Bel((\kappa \ominus \varphi) \ominus \psi) \\
\Leftrightarrow\ & [\![\kappa \ominus \varphi \vee \psi]\!] = [\![(\kappa \ominus \varphi) \ominus \psi]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} = [\![\kappa \ominus \varphi]\!] \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\} && \text{(Prop. 3.41)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \\
& = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\} && \text{(Prop. 3.41)}
\end{aligned}$$

Case $\kappa(\neg\varphi \wedge \neg\psi) = 0$:
In case that $\neg\varphi \wedge \neg\psi$ is assigned to rank 0, we know that the minimal models of $\neg\varphi \wedge \neg\psi$ must be included in $[\![\kappa]\!]$. Moreover, we can conclude that also the minimal models of $\neg\varphi$ and $\neg\psi$ are included in $[\![\kappa]\!]$, since each model of $\neg\varphi \wedge \neg\psi$ is also a model of $\neg\varphi$ and $\neg\psi$:

$$\begin{aligned}
& [\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\} \\
\Leftrightarrow\ & [\![\kappa]\!] = [\![\kappa]\!]
\end{aligned}$$

Thus, in case $\kappa(\neg\varphi \wedge \neg\psi) = 0$ the above-stated equality holds trivially.

Case $\kappa(\neg\varphi \wedge \neg\psi) > 0$:
Since $\neg\varphi \wedge \neg\psi$ is assigned to rank greater than 0, we know that its minimal models are not included in $[\![\kappa]\!]$. Thus,

$$[\![\kappa]\!] \cup \min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} = [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\}$$

can only hold, if

$$\min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\}.$$

Since we know from Lem. 2.55 that the minimal models of a conjunction $\neg\psi \wedge \neg\psi$ equals the intersection of the minimal models of $\neg\varphi$ and $\neg\psi$, if the intersection is not empty, we can conclude that the above-stated subset relation holds.

$$\min\{[\![\neg\varphi \wedge \neg\psi]\!], \preceq_\kappa\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\}$$

$$\begin{aligned}
&\Leftrightarrow \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} \\
&\quad \subseteq \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\psi]\!], \preceq_{\kappa\ominus\varphi}\} && \text{(Lem. 2.55)} \\
&\Leftrightarrow \min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} \subseteq \min\{[\![\neg\varphi]\!], \preceq_\kappa\}
\end{aligned}$$

In conclusion, we showed that in both cases $Bel(\kappa \ominus \varphi \vee \psi) \equiv Bel((\kappa \ominus \varphi) \ominus \psi)$ holds, if we assume the intersection of the minimal models of $\neg\varphi$ and $\neg\psi$ not to be empty. □

As stated in Th. 4.26, we know that the posterior beliefs after contracting κ with $\varphi \vee \psi$ or φ and ψ subsequently are equivalent, if the minimal models falsifying φ and ψ are not disjunct. This restriction is not necessary, but sufficient in order to satisfy the equivalence stated in **(DFPes-5)**$_\mathcal{L}$, since this way it is guaranteed that the same interpretations are added to the most plausible interpretations. Without this further restriction it would be possible that subsequently contracting φ and ψ only adds models of $\neg\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ to rank 0, which in fact would still allow to infer $\varphi \vee \psi$. We illustrate the sufficiency of $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} \neq \emptyset$ (Th. 4.26) in Ex. 4.8.

Example 4.8. *In this example we illustrate why the antecedence of the implication stated in Th. 4.26 is necessary to guarantee that the beliefs after contracting $\varphi \vee \psi$ are equivalent to those after contracting φ and ψ subsequently. For this, we consider the OCF κ in Tab. 28 and the formulas $\varphi \equiv p$ and $\psi \equiv f$, where*

$$\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} = \emptyset.$$

When we first contract p from κ by means of a minimal change c-contraction, we know that the ranks of all interpretations satisfying p remain unchanged, whereas the ranks of all models of $\neg p$ are decreased by $\gamma^- = \kappa(p) - \kappa(\neg p)$, such that there exist models of $\neg\varphi$ with rank 0 in the posterior OCF. The resulting OCF $\kappa \ominus \varphi$ is also given in Tab. 28. When we contract f from $\kappa \ominus \varphi$ afterwards, we again remain the ranks of all models of f unchanged, while decreasing the ranks of all models of $\neg f$ by $\gamma^- = \kappa(f) - \kappa(\neg f)$, such that there exist models of $\neg f$ in the posterior OCF. The result of forgetting p and f subsequently is given as $\kappa_c^\circ \ominus \psi$ in Tab. 28. The corresponding posterior beliefs are then given by

$$\begin{aligned}
Bel(\kappa_c^\circ \ominus \psi) &\equiv Th(\{p\overline{b}f, pbf, \overline{p}\overline{b}f, \overline{p}bf, pb\overline{f}, p\overline{b}\overline{f}\}) && \text{(Prop. 2.38)} \\
&\equiv Th([\![p \vee f]\!]) \\
&\equiv Cn(\bigvee_{\omega \in [\![p \vee f]\!]} \omega) && \text{(Lem. 2.23)} \\
&\equiv Cn(p \vee f) \models p \vee f.
\end{aligned}$$

Since contracting p and f subsequently only guarantees that p and f can no longer be inferred by the posterior beliefs, it can still be possible to infer $p \vee f$ afterwards. In case of the above-stated contractions, the posterior beliefs are even equivalent to $Cn(p \vee f)$, and therefore clearly differ from the beliefs of $\kappa \ominus p \vee f$

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{p}\overline{b}\overline{f}$, $\overline{p}b\overline{f}$
2	-
1	$\overline{p}\overline{b}f$, $\overline{p}bf$, $pb\overline{f}$, $p\overline{b}\overline{f}$
0	$p\overline{b}f$, pbf

$\kappa \ominus \varphi \vee \psi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	-
1	$\overline{p}\overline{b}f$, $\overline{p}bf$, $pb\overline{f}$, $p\overline{b}\overline{f}$
0	$p\overline{b}f$, pbf, $\overline{p}\overline{b}\overline{f}$, $\overline{p}b\overline{f}$

$\kappa \ominus \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$\overline{p}\overline{b}\overline{f}$, $\overline{p}b\overline{f}$
1	$pb\overline{f}$, $p\overline{b}\overline{f}$
0	$p\overline{b}f$, pbf, $\overline{p}\overline{b}f$, $\overline{p}bf$

$\kappa_c^\circ \ominus \psi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	-
1	$\overline{p}\overline{b}\overline{f}$, $\overline{p}b\overline{f}$
0	$p\overline{b}f$, pbf, $\overline{p}\overline{b}f$, $\overline{p}bf$, $pb\overline{f}$, $p\overline{b}\overline{f}$

Table 28: Contractions of $\varphi \equiv p$, $\psi \equiv f$ and $\varphi \vee \psi$ in κ. Top left: OCF κ over signature Σ_{Tweety}. Top right: Minimal change c-contraction $\kappa \ominus \varphi \vee \psi$ with parameters $\gamma^- = -3$ and $\gamma^+ = \kappa_0 = 0$. Bottom left: Minimal change c-contraction $\kappa \ominus \varphi$ with parameters $\gamma^- = -1$ and $\gamma^+ = \kappa_0 = 0$. Bottom right: Minimal change c-contraction $\kappa_c^\circ \ominus \psi$ with parameters $\gamma^- = -1$ and $\gamma^+ = \kappa_0 = 0$, where κ_c° corresponds to $\kappa \ominus \varphi$.

(Tab. 28), which do not infer $p \vee f$:

$$\begin{aligned} Bel(\kappa \ominus p \vee f) &\equiv Th([\![\kappa \ominus p \vee f]\!]) && \text{(Prop. 2.38)} \\ &\equiv Th(\{p\overline{b}f, pbf, \overline{p}\overline{b}\overline{f}, \overline{p}b\overline{f}\}) \not\models p \vee f \end{aligned}$$

The reason for the different posterior beliefs lays in the disjunct minimal models of $\neg p$ and $\neg f$. All minimal models of $\neg p$ in κ are models of f. Thus, contracting κ with p only adds models of $\neg p \wedge f$ to the most plausible interpretations. Furthermore, the contraction with p decreases the ranks of the models of $\neg p$, but since the minimal models of $\neg f$ do not contain models of $\neg p$ they are not affected by this. In addition to this, the rank difference between the minimal models of $\neg f$ and $\neg f \wedge \neg p$ is greater than the rank difference of p and $\neg p$, such that the minimal models of f after the contraction with p do not change. Thus, when contracting with f afterwards, only models of $\neg f \wedge p$ are added to the prior most plausible interpretations. In conclusion, we know that no models of $\neg p \wedge \neg f$ were added to rank 0, since the minimal models *of p and f were disjunct.*

Summary of the examinations of (DFPes-1)$_\mathcal{L}$-(DFPes-6)$_\mathcal{L}$. Finally, we want to summarize the results of our examinations on minimal change c-contractions and the postulates **(DFPes-1)$_\mathcal{L}$-(DFPes-6)$_\mathcal{L}$** in Th. 4.27.

Theorem 4.27. *Let $\ominus$ be a minimal change c-contraction, then the following holds:*

- $\ominus$ *satisfies **(DFPes-1)$_\mathcal{L}$**,*
- $\ominus$ *satisfies **(DFPes-2)$_\mathcal{L}$**, if $\kappa \sqsubseteq \kappa'$,*
- $\ominus$ *satisfies **(DFPes-3)$_\mathcal{L}$**,*
  only if $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} = \min\{[\![\neg\psi]\!], \preceq_\kappa\}$ *or* $\kappa(\neg\varphi) = 0 = \kappa(\neg\psi)$ *or* $\varphi \not\models \psi$,
- $\ominus$ *satisfies **(DFPes-4)$_\mathcal{L}$**,*
  only if $\kappa(\neg\varphi) = \kappa(\neg\psi)$ *or* $\kappa(\neg\varphi) = 0$ *or* $\kappa(\neg\psi) = 0$,
- $\ominus$ *satisfies **(DFPes-5)$_\mathcal{L}$**, if* $\min\{[\![\neg\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\neg\psi]\!], \preceq_\kappa\} \neq \emptyset$,
- $\ominus$ *satisfies **(DFPes-6)$_\mathcal{L}$**.*

For proofs and explanations of the stated relations, we refer to the elaborations above. In summary, we showed that minimal change c-contractions are capable of satisfying **(DFPes-1)$_\mathcal{L}$** and **(DFPes-6)$_\mathcal{L}$**. This goes back to the fact that minimal change c-contractions also satisfy **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40). For the remaining forgetting postulates **(DFPes-2)$_\mathcal{L}$-(DFPes-5)$_\mathcal{L}$**, we showed that assuming such c-contractions that only induce minimal change to the prior beliefs is not sufficient to satisfy them. However, we further showed that these postulate can be satisfied under further assumptions. These assumptions mostly argue about the relation between the minimal models, either in general or with respect to the formulas we like to forget. Another important aspect that occurs within these additional assumptions is the question, whether the formula we want to forget can be inferred by the prior beliefs in the first place. This concerns the postulates **(DFPes-3)$_\mathcal{L}$** and **(DFPes-4)$_\mathcal{L}$** and is stated by assuming the formulas to be assigned to rank 0.

While examining the forgetting postulates for minimal change c-contraction, we noted some connections between the forgetting and the AGM contraction postulates. Since the AGM theory is very fundamental in the domain of knowledge representation, we want to examine further connections in the following.

4.2.4 Further Connections between AGM Contractions and Forgetting

In the previous section, we already elaborated some of the connections between the AGM contraction and the forgetting postulates in the context of minimal change c-contractions. In this section, we briefly examine more general connections that are independent of the chosen belief change operator. By means of minimal change c-contractions, we already showed that the AGM postulates **(AGMes-1)**-**(AGMes-7)** (see Section 2.3 or Appendix A.1) are not sufficient for a belief change operator to also satisfy the forgetting postulates. However, it can be shown that the forgetting postulates imply all of the AGM contraction postulates except for **(AGMes-2)** and the recovery postulate **(AGMes-4)**. We state this implication in Th. 4.28 and intuitively explain it for each of the postulates afterwards.

Theorem 4.28. *Let $\circ_f^{\mathcal{L}}$ be a belief change operator satisfying **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$, then $\circ_f^{\mathcal{L}}$ also satisfies all AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-6)**, except for **(AGMes-2)** and **(AGMes-4)**.*

Proof of Th. 4.28.

(AGMes-1): $\circ_f^{\mathcal{L}}$ satisfies **(AGMes-1)**, since it is equivalent to **(DFPes-1)**$_{\mathcal{L}}$.

(AGMes-3): $\circ_f^{\mathcal{L}}$ satisfies **(AGMes-3)**, since it is equivalent to **(DFPes-6)**$_{\mathcal{L}}$.

(AGMes-5): Let φ and ψ be formulas with $\varphi \equiv \psi$, then the following holds:

$$\begin{aligned}
Bel(\Psi \circ_f^{\mathcal{L}} \varphi) &\equiv Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-3)}_{\mathcal{L}} \\
&\equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) && \text{(Prop. 4.8)} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} \psi) && \textbf{(DFPes-3)}_{\mathcal{L}}
\end{aligned}$$

(AGMes-6):

$$\begin{aligned}
Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) &\equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \varphi \wedge \psi) && \textbf{(DFPes-3)}_{\mathcal{L}} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee (\varphi \wedge \psi)) && \textbf{(DFPes-5)}_{\mathcal{L}} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \textbf{(AGMes-5)} \\
&\models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \vee Bel(\Psi \circ_f^{\mathcal{L}} \psi)
\end{aligned}$$

(AGMes-7): We prove that a forgetting operator $\circ_f^{\mathcal{L}}$ satisfies the conclusion of **(AGMes-7)** generally and thus, especially if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \not\models \varphi$.

$$\begin{aligned}
Bel(\Psi \circ_f^{\mathcal{L}} \varphi) &\models Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) && \textbf{(DFPes-1)}_{\mathcal{L}} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) && \textbf{(DFPes-5)}_{\mathcal{L}} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} (\varphi \vee \psi) \vee (\varphi \wedge \psi)) && \textbf{(AGMes-5)} \\
&\equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) \circ_f^{\mathcal{L}} \varphi \wedge \psi) && \textbf{(DFPes-5)}_{\mathcal{L}} \\
&\equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) && \textbf{(DFPes-3)}_{\mathcal{L}}
\end{aligned}$$

□

Since **(AGMes-1)** is equivalent to **(DFPes-1)**$_{\mathcal{L}}$ it follows trivially from the forgetting postulates. **(AGMes-3)** concludes from the forgetting postulates, because forgetting a certain information always results in beliefs that are not able to infer this information anymore. Since this holds in general, it especially holds for non-tautologous formulas.

(AGMes-5) holds, since forgetting two equivalent pieces of information always results in the same knowledge, due to the fact that forgetting two pieces of information, where one is already included in the other, results in the same beliefs as just forgetting the more general piece of information. The equivalence states that they are mutually included, and therefore it is sufficient to forget either of them, since they will both result in the same beliefs.

The implication of **(AGMes-6)** can be traced back to the fact that forgetting two pieces of information together ($\varphi \wedge \psi$) is equivalent to forgetting them simultaneously ($\varphi \vee \psi$). Furthermore, we know that simultaneously forgetting two pieces of information is equivalent to forgetting them separately and accepting those beliefs both resulting belief sets agree on. Finally, the beliefs that are accepted this way can at least infer those that are believed after forgetting one information or the other.

After forgetting a certain information φ, we can also infer all the beliefs that are inferable, if we would forget it together with another information ψ, i.e. $\varphi \wedge \psi$. Since this generally holds for the concept of forgetting, it especially holds if we assume that φ can no longer be inferred after forgetting both information together. Thus, we know that the forgetting postulates also imply **(AGMes-7)**.

The only contraction postulates that are not implied by the forgetting postulates are **(AGMes-2)** and **(AGMes-4)**. The fact that the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ do not imply **(AGMes-2)** means that it is possible for a belief change operator $\circ_f^\mathcal{L}$ satisfying these postulates to reduce the prior beliefs, even if the formula we want to forget could not be inferred by them. Since the recovery postulate **(AGMes-4)** is not implied by **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ either, we know that after forgetting a formula φ, the prior beliefs cannot be restored by adding φ and the posterior beliefs. This means that forgetting with an operator $\circ_f^\mathcal{L}$ satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ potentially removes more information than those that can be inferred by φ. The fact that such an forgetting operator $\circ_f^\mathcal{L}$ does not necessarily have to satisfy **(AGMes-2)** and **(AGMes-4)** clearly shows that $\circ_f^\mathcal{L}$ does not have to comply with the concept of minimal change. This can be especially interesting in the context of iterated belief change, where inducing minimal changes is not always a desired behaviour, since it often requires dismissing further conditional relations (see [DP97]).

4.2.5 On the Equivalence of Minimal Change C-Contractions and Delgrande's Forgetting Approach

Finally, we show that minimal change c-contractions not only fail at satisfying most of the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ without further assumptions (Th. 4.27), but neither do produce posterior beliefs equivalent to the result of Delgrande's forgetting approach as presented in Section 3.1. Instead of directly comparing the results of Delgrande's approach $\mathcal{F}(Bel(\kappa), P)$ to those of minimal change c-contractions $Bel(\kappa \ominus \varphi)$, we want to make use of the fact that the marginalization of an OCF results in beliefs $Bel(\kappa_{|\Sigma \setminus P})$ equivalent to Delgrande's forgetting (Th. 4.1). This way, the comparison is more straight-forward, because both the marginalization and minimal change c-contraction argue about OCFs. From Prop. 2.38 we know that the beliefs after a contraction and a marginalization are equivalent, if the posterior most plausible interpretations are the same. Since the marginalization yields a posterior OCF, which is only defined over a subsignature of the prior, we will lift the posterior OCF back to the original signature according to the definition of the unique lifting given in Def. 3.28. Thus, we say that a minimal change c-contraction $\kappa \ominus \varphi$ and a marginalization $\kappa_{|\Sigma \setminus P}$ result in equivalent belief, if and only if $[\![\kappa \ominus \varphi]\!] = [\![(\kappa_{|\Sigma \setminus P})_{\uparrow \Sigma}]\!]$. In addition to this consideration, it is necessary to define how the marginalized signature P and the forgotten formula φ relate to each other. For this, we recall the notions behind the marginalization from Section 3.2.1, which is among others the focussing on relevant aspects $\Sigma \setminus P$, such that our beliefs do not contain any information about the irrelevant aspects P anymore. This means that we are not capable of inferring any propositions about the forgotten signature elements P afterwards, except for tautologies and those that directly conclude from the beliefs of the marginalized OCF $\kappa_{|\Sigma \setminus P}$. Note that at this point we already assume a consecutively performed lifting of the marginalized OCF back to the original signature Σ. When we restrict the marginalization to single signature elements $\rho \in \Sigma$, this concludes that it is not sufficient to perform a single contraction in order to capture the notions of the marginalization and result in equivalent beliefs. For this it is necessary that we contract both the positive literal ρ, as well as its negative counterpart $\neg\rho$. Otherwise, it could be possible to infer either ρ or $\neg\rho$ afterwards. At this point, we make use of the fact that signature elements can also be viewed as atomic propositions (Def. 2.2). Therefore, we say that a marginalization of a single signature element $\rho \in \Sigma$ should be described by contracting the corresponding positive and negative literal ρ and $\neg\rho$ consecutively:

$$Bel((\kappa \ominus \rho) \ominus \neg\rho) \equiv Bel((\kappa_{|\Sigma \setminus \{\rho\}})_{\uparrow \Sigma}) \tag{4.4}$$

Notice that the order in which ρ and $\neg\rho$ are contracted is irrelevant, since we know that either a model of ρ or $\neg\rho$ must be assigned to rank 0 (Prop. 2.35), and therefore at least one of the contractions will not affect κ, due to **(AGMes-1)** and **(AGMes-2)** (Lem. 4.29).

Lemma 4.29. *Let κ be an OCF over signature Σ, $\rho \in \mathcal{L}_\Sigma$ an atomic formula, and $\ominus$ a minimal change c-contraction, then the following holds:*

$$(Bel((\kappa \ominus \rho) \ominus \neg\rho) \equiv Bel(\kappa \ominus \rho)) \text{ or } (Bel((\kappa \ominus \rho) \ominus \neg\rho) \equiv Bel(\kappa \ominus \neg\rho))$$

However, it can be shown that the equivalence stated in Eq. 4.4 does not hold in general. Therefore, it is not possible to relate minimal change c-contractions and marginalizations by means of forgetting the literals ρ and $\neg\rho$ consecutively. We formalize this relation in Prop. 4.30 and prove it by means of a counterexample.

Proposition 4.30. *Let κ be an OCF over signature Σ, $\rho \in \Sigma$ a signature element or atomic formula, respectively, and $\ominus$ a minimal change c-contraction, then*

$$Bel((\kappa \ominus \rho) \ominus \neg\rho) \equiv Bel((\kappa_{|\Sigma \setminus \{\rho\}})_{\uparrow \Sigma})$$

does not hold in general.

Proof of Prop. 4.30. We prove Prop. 4.30 by giving a concrete counterexample to the there stated equivalence. For this, let κ be the OCF over signature $\Sigma_{Tweety} = \{p, b, f\}$ as given in Tab. 29 below.

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	$p\overline{b}\overline{f}$, $pb\overline{f}$
1	pbf, $p\overline{b}f$, $\overline{p}bf$
0	$\overline{p}\overline{b}\overline{f}$,$\overline{p}b\overline{f}$, $\overline{p}\overline{b}f$

Table 29: OCF κ over signature Σ_{Tweety}.

In the following, let $\rho = p$, be the signature element and $\rho \equiv p$, $\neg\rho \equiv \neg p$ the literals we want to forget. The marginalization of p in κ (Def. 3.23) and the consecutively performed lifting to Σ_{Tweety} (Def. 3.28) are given in Tab. 30, and denoted by $\kappa_{|\Sigma_{Tweety}\setminus\{p\}}$ and $(\kappa_{|\Sigma_{Tweety}\setminus\{p\}})_{\uparrow\Sigma_{Tweety}}$.

$\kappa_{\vert\Sigma\setminus\{p\}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}\setminus\{p\}}$
∞	-
$\vdots$	-
2	-
1	bf
0	$\overline{b}\overline{f}$, $b\overline{f}$, $\overline{b}f$

$(\kappa_{\vert\Sigma\setminus\{p\}})_{\uparrow\Sigma_{Tweety}}(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
2	-
1	pbf, $\overline{p}bf$
0	$p\overline{b}\overline{f}$, $\overline{p}\overline{b}\overline{f}$ $pb\overline{f}$, $\overline{p}b\overline{f}$, $p\overline{b}f$, $\overline{p}\overline{b}f$

Table 30: Top: Marginalization of κ (Tab. 29) to the subsignature $\Sigma_{Tweety}\setminus\{p\}$. Bottom: Lifting of $\kappa_{|\Sigma\setminus\{p\}}$ to Σ_{Tweety}.

Next, we state the minimal change c-contraction of p and $\neg p$ in κ. First, we show that the first contraction $\kappa \ominus p$ does not affect κ, since $Bel(\kappa) \not\models p$. According to the definition of c-changes (Def. 3.35), we know that the ranks of all interpretations are shifted by means of parameters γ^+, γ^- and κ_0, where

$$\kappa \ominus \varphi\ (\omega) = \kappa(\omega) - \kappa_0 + \begin{cases} \gamma^+, & \text{if } \omega \models \varphi \\ \gamma^-, & \text{if } \omega \models \neg\varphi \end{cases}$$

holds for all formulas $\varphi \in \mathcal{L}_\Sigma$. Due to the definition of minimal change c-contractions (Def. 3.37), we know that the following holds for the parameters γ^+, γ^- and κ_0:

$$\begin{aligned}
\gamma^+ &= 0,\\
\gamma^- &= \min\{0, \kappa(\varphi) - \kappa(\neg\varphi)\},\\
\kappa_0 &= \gamma^- + \kappa(\neg\varphi).
\end{aligned}$$

For the OCF κ as given in Tab. 29 and the atomic formula p all of the parameters equal 0. Therefore, we know that

$$\kappa \ominus p\ (\omega) = \kappa(\omega) - \kappa_0 + \begin{cases} \gamma^+, & \text{if } \omega \models p \\ \gamma^-, & \text{if } \omega \models \neg p \end{cases} = \kappa(\omega) - 0 + \begin{cases} 0, & \text{if } \omega \models p \\ 0, & \text{if } \omega \models \neg p \end{cases} = \kappa(\omega)$$

holds for each $\omega \in \Omega_{\Sigma_{Tweety}}$. This further concludes $(\kappa \ominus p) \ominus \neg p = \kappa \ominus \neg p$. Thus, we only have to consider the minimal change c-contraction of $\neg p$ in the following, which is given in Tab. 31. Note that the parameters $\gamma^+, \gamma^-, \kappa_0$ are again determined by Def. 3.37 as already stated above.

$\kappa \ominus \neg p\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
2	-
1	$\overline{p}bf$, $p\overline{b}\overline{f}$, $pb\overline{f}$
0	$\overline{p}\overline{b}\overline{f}, \overline{p}b\overline{f}$, $\overline{p}\overline{b}f$, pbf, $p\overline{b}f$

Table 31: Posterior OCF $\kappa \ominus \neg p$ after contracting $\neg p$ in κ (Tab. 29) by means of a minimal change c-contraction with parameters $\gamma^+ = 0$, $\gamma^- = -1$, $\kappa_0 = 0$.

Given the resulting OCFs $(\kappa_{|\Sigma_{Tweety} \setminus \{p\}})_{\uparrow \Sigma_{Tweety}}$ and $\kappa \ominus \neg p$, we show that their beliefs are not equivalent, since the corresponding most plausible interpretations are not equal:

$$\begin{aligned}
& Bel((\kappa_{|\Sigma_{Tweety} \setminus \{p\}})_{\uparrow \Sigma_{Tweety}}) \equiv Bel(\kappa \ominus \neg p) \\
\Leftrightarrow\ & [\![(\kappa_{|\Sigma_{Tweety} \setminus \{p\}})_{\uparrow \Sigma_{Tweety}}]\!] = [\![\kappa \ominus \neg p]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\ & \{p\overline{b}\overline{f}, \overline{p}\overline{b}\overline{f} pb\overline{f}, \overline{p}b\overline{f}, p\overline{b}f, \overline{p}\overline{b}f\} = \{\overline{p}\overline{b}\overline{f}, \overline{p}b\overline{f}, \overline{p}\overline{b}f, pbf, p\overline{b}f\} && \lightning
\end{aligned}$$

In conclusion, we showed that the equivalence stated in Prop. 4.30 does not hold in general. $\square$

In the further course, we intuitively explain why the posterior beliefs after a marginalization of a signature element $\rho \in \Sigma$ cannot be expressed by means of contracting the corresponding positive and negative literal ρ and $\neg\rho$ as stated in Prop. 4.30 above. From Th. 4.3, we know that due to the marginalization all interpretations that are equivalent to those in $[\![\kappa]\!]$ with respect to the reduced signature

$\Sigma \setminus \{\rho\}$ are additionally assigned to rank 0 in the posterior OCF $\kappa_{|\Sigma\setminus\{\rho\}}$. On the other hand, a minimal change c-contraction only adds those minimal models to the prior most plausible interpretations $[\![\kappa]\!]$ that falsify the contracted literal. However, due to the most plausible interpretations after a lifting (Th. 4.3 and Cor. 4.4), i.e.

$$[\![(\kappa_{|\Sigma\setminus\{\rho\}})_{\uparrow\Sigma}]\!] = \{\omega \in \Omega_\Sigma \mid \text{there exists } \omega' \in [\![\kappa]\!] \text{ with } \omega \equiv_{\Sigma\setminus\{\rho\}} \omega'\},$$

we know that for each $\omega' \in [\![(\kappa \ominus \rho) \ominus \neg\rho]\!]$ all $\omega \in \Omega_\Sigma$ with $\omega \equiv_{\Sigma\setminus\{\rho\}} \omega'$ must be included in the posterior most plausible interpretations as well. This clearly cannot be guaranteed when just adding the minimal models of ρ or $\neg\rho$ to $[\![\kappa]\!]$ as seen in the proof of Prop. 4.30. Therefore, we know that the beliefs after a marginalization of ρ are not generally equivalent to those after contracting ρ and $\neg\rho$.

Summary. In summary, we extended the forgetting postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4) as originally stated by Delgrande [Del17] such that they are applicable to arbitrary belief change operators. This allows us to make use of them for further research presented in this thesis and future work. We showed that arbitrary c-contractions do not satisfy the forgetting postulates except for **(DFPes-6)**$_{\mathcal{L}}$, mostly due to the changes they can induce to the prior beliefs. Because of this, we examined the forgetting postulates for minimal change c-contraction and showed that contracting knowledge with respect to the minimal change paradigm is not sufficient to satisfy the forgetting postulates either. However, we elaborated further conditions that are necessary to satisfy the remaining postulates. These conditions emphasize that arguing about the most plausible interpretations is often insufficient and that the order of the remaining interpretations plays an essential role, too. Thus, the concept of refinement is of importance when it comes to comparing the beliefs of multiple epistemic states. Furthermore, we elaborated further connections between the generalized forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ and the AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-7)** (see Section 2.3 or Appendix A.1). We showed that **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ imply most of the contraction postulates, namely **(AGMes-1)**,**(AGMes-3)** and **(AGMes-5)**-**(AGMes-7)**. Thus, only **(AGMes-2)** and **(AGMes-4)** are not implied by **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$. However, both of the contraction postulates are not excluded by the forgetting postulates either. The fact that especially **(AGMes-2)** and **(AGMes-4)** are not implied, shows that a forgetting operator according to **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ does not necessarily have to perform belief changes according minimal change paradigm. This is of particular interest in the context of iterated belief change, where the concept of minimal change is not always favourable. In the examination of the connection between the forgetting and contraction postulates, we noticed some implicit properties implied by the forgetting postulates, such as $Bel(\kappa \circ_f^{\mathcal{L}} \varphi \vee \psi) \equiv Bel(\kappa \circ_f^{\mathcal{L}} \varphi \wedge \psi)$, which indicate that the postulates as formulated in this work might not be appropriate to describe the concept of forgetting, but rather form a first attempt of formalizing it. A more detailed elaboration of the forgetting postulates is subject of Section 4.4. Finally, we examined how minimal change c-contractions relate to Delgrande's general forgetting approach. For this we compared the posterior beliefs of minimal change c-contraction to those of the marginalization, for which we already know that they are equivalent to the result of Delgrande's approach (Th. 4.1). We showed that minimal change c-contractions do not result in beliefs equivalent to Delgrande's approach in general, when trying to capture the notions of marginalizing a single signature element by contracting the corresponding literals consecutively.

4.3 Revision

In this section, we discuss the concept of revision in the context of forgetting. For this, we consider revisions in the sense of c-revisions [KI04] as already described in Section 3.2.3 and examine their forgetting properties by means of the previously stated forgetting postulates for epistemic states **(DFPes-1)**$_\mathcal{L}$ - **(DFPes-6)**$_\mathcal{L}$. Since the forgetting aspect of revisions is of implicit nature, we will elaborate an explicit representation of it first, and then examine how it relates to the concept of forgetting. Afterwards, we examine which of the properties valid for the implicit forgetting of c-revisions can be transferred to the revision itself. Note that even if we focus on c-revisions in the further examinations, we will formulate our results as general as possible. Therefore, some of the results will refer to revision operators satisfying **(AGMes$*$1)**-**(AGMes$*$6)** instead. However, this always includes propositional c-revisions as well, since we know that they satisfy **(AGMes$*$1)**-**(AGMes$*$6)** (Prop. 3.47). Finally, we examine more general relations between the concepts of forgetting and revision, by comparing the corresponding postulates to each other. For this, we concretely investigate which revision postulates are also satisfied by belief change operators satisfying the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$. For the revision postulates we consider the revision postulates for epistemic states **(AGMes$*$1)**-**(AGMes$*$6)** on the one hand, and the postulates for iterated revision **(DP1)**-**(DP4)** on the other (see Section 2.3 or Appendix A.1).

4.3.1 C-Revisions as Forgetting Operators

The forgetting described by c-revisions is of implicit nature, since its success postulate only describes that we should be able to infer a certain conditional $(\psi|\varphi)$ after the revision with $(\psi|\varphi)$. The implicit forgetting is described by the removal of knowledge contradictory to $(\psi|\varphi)$, which in turn guarantees the fulfilment of the c-revision's success postulate. In concrete terms, this is done by adjusting the models of $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ in such a way that $\varphi \wedge \psi$ is more plausible than $\varphi \wedge \neg\psi$ in the resulting epistemic state. If we further want to describe the implicit forgetting of c-revisions explicitly, we can use the fact that c-revisions are revisions in the sense of AGM (Prop. 3.47). For the latter we know, according to the **(Levi equivalence)** (see Section 2.3), that the revision can also be represented as successively applying a contraction and an expansion to a prior epistemic state. Since the c-revision is defined over OCFs, the contraction and expansion must also be defined over OCFs, otherwise it would not be possible to apply the different belief change operators consecutively. From this we can conclude that the implicit forgetting of a c-revision can be explicitly described by an AGM contraction over OCFs. We already formulated this connection between c-revisions and contractions satisfying

(AGMes-1)-**(AGMes-7)** in Prop. 3.52. Therefore, we know that the implicit forgetting, i.e. the contraction of beliefs contradicting the information we revise with, can also be represented by minimal change c-contractions (Lem. 4.31).

Lemma 4.31. *Let κ be an OCF over signature Σ, $\varphi \in \mathcal{L}_\Sigma$ a formula, $\circledast$ a propositional c-revision, and $\ominus$ a minimal change c-contraction, then the following holds:*

$$Bel(\kappa \circledast \varphi) \equiv Bel((\kappa \ominus \neg\varphi) \circledast \varphi)$$

Lem. 4.31 directly concludes from Prop. 3.52, since we know that minimal change c-contractions satisfy the contraction postulates **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40). Thus, we know that the implicit forgetting of c-revisions inherits the forgetting properties of minimal change c-contractions, as elaborated and discussed in detail in Section 4.2. Therefore, we will investigate in the following how the forgetting properties of minimal change c-contractions can be transferred to c-revisions, and how they change when the OCF's beliefs are expanded after the contraction.

Moreover, in order to examine how the elaborated properties of minimal change c-contractions from Section 4.2 behave for the c-revisions, we need to know how the most plausible interpretations of an OCF are changed by them. For this we make use of Lem. 4.31 above, which states that a c-revision can be expressed by means of a minimal change c-contraction and a subsequently performed c-revision, and the posterior most plausible interpretations after a minimal change c-contraction (Def. 3.37). The minimal change c-contraction with $\neg\varphi$ extends the previous most plausible interpretations by the minimal models of φ (Prop. 3.41). Thus, there must exist models of φ that are assigned to rank 0 after contracting $\neg\varphi$, and only the models of $\neg\varphi$ have to be removed from rank 0 by the following expansion with φ in order to fulfil the c-revisions success postulate. Notice that this holds for operators satisfying **(AGMes-1)**-**(AGMes-7)** and **(AGMes$*$1)**-**(AGMes$*$6)** in general. Thus, we will first state and prove the general case in Prop. 4.32 below, and afterwards state this relation explicitly for c-revision and minimal change c-contractions in Lem. 4.33.

Proposition 4.32. *Let Ψ be an epistemic state with faithfully assigned total preorder $\preceq_\Psi$ and $\varphi \in \mathcal{L}_\Sigma$ a formula. Furthermore, let $*$ be a belief change operator satisfying* ***(AGMes$*$1)**-**(AGMes$*$6)*** *and $-$ a belief change operator satisfying* ***(AGMes-1)**-**(AGMes-7)****, then the following holds:*

$$[\![\Psi * \varphi]\!] = [\![\Psi - \neg\varphi]\!] \setminus [\![\neg\varphi]\!].$$

Proof of Prop. 4.32.

$$
\begin{aligned}
& [\![\Psi * \varphi]\!] = [\![\Psi - \neg\varphi]\!] \setminus [\![\neg\varphi]\!] \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\Psi\} = [\![\Psi - \neg\varphi]\!] \setminus [\![\neg\varphi]\!] && \text{(Th. 3.46)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\Psi\} = ([\![\Psi]\!] \cup \min\{[\![\varphi]\!], \preceq_\Psi\}) \setminus [\![\neg\varphi]\!] && \text{(Th. 3.42)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\Psi\} = ([\![\Psi]\!] \setminus [\![\neg\varphi]\!]) \cup \min\{[\![\varphi]\!], \preceq_\Psi\} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\Psi\} = \min\{[\![\varphi]\!], \preceq_\Psi\} \cup \begin{cases} \emptyset, & \text{if } [\![\Psi]\!] \cap [\![\varphi]\!] = \emptyset \\ \min\{[\![\varphi]\!], \preceq_\Psi\}, & \text{if } [\![\Psi]\!] \cap [\![\varphi]\!] \neq \emptyset \end{cases}
\end{aligned}
$$

$\Leftrightarrow \min\{[\![\varphi]\!], \preceq_\Psi\} = \min\{[\![\varphi]\!], \preceq_\Psi\}$

□

Lemma 4.33. *Let κ be an OCF over signature Σ and $\varphi \in \mathcal{L}_\Sigma$ a formula. Furthermore, let $\circledast$ be a c-revision and $\ominus$ a minimal change c-contraction, then the following holds:*

$$[\![\kappa \circledast \varphi]\!] = [\![\kappa \ominus \neg\varphi]\!] \setminus [\![\neg\varphi]\!].$$

Lem. 4.33 directly concludes from Prop. 4.32, since c-revision satisfy **(AGMes$*$1)**-**(AGMes$*$6)** (Prop. 3.47) and minimal change c-contraction satisfy **(AGMes-1)**-**(AGMes-7)** (Prop. 3.40).

After we have shown above that the implicit forgetting of c-revisions can be represented by minimal change c-contractions and how the most plausible interpretations change due to them, we will further examine how the relations of minimal change c-contractions and the forgetting postulates **(DFPes-1)$_\mathcal{L}$**-**(DFPes-6)$_\mathcal{L}$** as stated in Th. 4.27 behave for c-revisions.

At first we examine the postulates **(DFPes-1)$_\mathcal{L}$** and **(DFPes-6)$_\mathcal{L}$**, since these are the only postulates satisfied by minimal change c-contractions without further assumptions. **(DFPes-1)$_\mathcal{L}$** cannot be satisfied by revisions satisfying **(AGMes$*$1)**-**(AGMes$*$6)** in general, because a revision expands the prior beliefs by definition, which contradicts the idea behind the first forgetting postulate, in which a forgetting operator never expands the prior beliefs (Prop. 4.34).

Proposition 4.34. *Let Ψ be an epistemic state with faithfully assigned total preorder $\preceq_\Psi$, $\varphi \in \mathcal{L}_\Sigma$ a formula, and $*$ a belief change operator satisfying **(AGMes$*$1)**-**(AGMes$*$6)**, then $*$ satisfies*

$$Bel(\Psi) \models Bel(\Psi * \varphi), \qquad \textbf{(DFPes-1)}_\mathcal{L}$$

only if $Bel(\Psi) \models \varphi$.

Proof of Prop. 4.34.

$$\begin{aligned}
& Bel(\Psi) \models Bel(\Psi * \varphi) & \\
\Leftrightarrow\ & [\![\Psi]\!] \subseteq [\![\Psi * \varphi]\!] & \text{(Def. 2.12)}\\
\Leftrightarrow\ & [\![\Psi]\!] \subseteq \min\{[\![\varphi]\!], \preceq_\Psi\} & \text{(Th. 3.46)}\\
\Rightarrow\ & [\![\Psi]\!] \subseteq [\![\varphi]\!] & \\
\Rightarrow\ & Bel(\Psi) \models \varphi & \text{(Def. 2.12)}
\end{aligned}$$

□

(DFPes-1)$_\mathcal{L}$ would only be satisfied in the trivial case, in which the formula we revise with could already be concluded by the prior belief set. In this case, the prior and posterior beliefs would be equivalent, since the c-revision would not affect the most plausible interpretations. In Lem. 4.35, we explicitly state the relation of revisions and **(DFPes-1)$_\mathcal{L}$** for propositional c-revisions.

Lemma 4.35. *Let κ be an OCF over signature Σ, $\varphi \in \mathcal{L}_\Sigma$ a formula, and $\circledast$ a propositional c-revision, then $\circledast$ satisfies*

$$Bel(\kappa) \models Bel(\kappa \circledast \varphi), \qquad \textbf{(DFPes-1)}_{\mathcal{L}}$$

only if $Bel(\kappa) \models \varphi$.

Lem. 4.35 directly concludes from Prop. 4.34, since propositional c-revision satisfy **(AGMes∗1)**-**(AGMes∗6)**. However, the fact that c-revisions do not satisfy **(DFPes-1)**$_{\mathcal{L}}$ does not contradict the idea of regarding c-revisions, or the concept of revision in general, as a kind of forgetting on an intuitive level. Firstly, the reduction of the belief set is not the actual intention of a revision, since it only reduces the prior beliefs to guarantee that there are no conclusions contradicting the newly added knowledge. Thus, when we argue about revisions in the sense of forgetting, we should focus on the contraction of the contradicting conclusions performed by the revision. Secondly, we know that the implicit forgetting performed by c-revisions satisfies **(DFPes-1)**$_{\mathcal{L}}$, since any operator satisfying **(AGMes-1)**-**(AGMes-7)** can be used to represent the implicit forgetting (Prop. 3.52) and **(DFPes-1)**$_{\mathcal{L}}$ is equivalent to **(AGMes-1)**. We follow a similar argumentation for **(DFPes-6)**$_{\mathcal{L}}$. This postulate is not satisfied by c-revisions either, because it also contradicts the underlying success postulate of c-revisions. However, one must note at this point that the implicit forgetting of c-revisions is not applied to φ itself, but to $\neg\varphi$, whereas **(DFPes-6)**$_{\mathcal{L}}$ also refers to φ. So in order to capture the success of the implicit forgetting accurately, we have to consider **(DFPes-6)**$_{\mathcal{L}}$ for $\neg\varphi$ instead. In this case we know that c-revisions satisfy **(DFPes-6)**$_{\mathcal{L}}$ (Lem. 4.36). Therefore, in the context of forgetting, revising with a formula φ can be regarded as forgetting $\neg\varphi$.

Lemma 4.36. *Let κ be an OCF over signature Σ and $\varphi \in \mathcal{L}_\Sigma$ a formula, then $\kappa \circledast \varphi \not\models \neg\varphi$ holds for each c-revision $\kappa \circledast \varphi$.*

In the following, we examine **(DFPes-2)**$_{\mathcal{L}}$ in the context of c-revisions, which states that revising with the same formula in two OCFs κ and κ' results in posterior beliefs $Bel(\kappa \circledast \varphi) \models Bel(\kappa' \circledast \varphi)$, if the prior beliefs of κ' could also be inferred by the prior beliefs of κ. We show that the subsequently performed expansion after the contraction has no effect on the fulfilment of **(DFPes-2)**$_{\mathcal{L}}$, and that it only depends on the implicitly performed forgetting. Thus, we show that **(DFPes-2)**$_{\mathcal{L}}$ holds under the same conditions as for minimal change c-contractions. Additionally, we prove that it is even possible to maintain the refinement relation of the two prior OCFs after the revision, when choosing the parameters appropriately.

Since minimal change c-contractions form the implicit forgetting of c-revisions, we can trace back the consequence of **(DFPes-2)**$_{\mathcal{L}}$ to the most plausible interpretations of κ and κ' after performing the contractions $\kappa \ominus \neg\varphi$ and $\kappa' \ominus \neg\varphi$:

$$\begin{aligned}
& Bel(\kappa \ominus \neg\varphi) \models Bel(\kappa' \ominus \neg\varphi) \\
\Leftrightarrow\ & [\![\kappa \ominus \neg\varphi]\!] \subseteq [\![\kappa' \ominus \neg\varphi]\!] && \text{(Prop. 2.41)} \\
\Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq [\![\kappa]\!] \cup \min\{[\![\varphi]\!], \preceq_\kappa\} && \text{(Prop. 3.41)}
\end{aligned}$$

For minimal change c-contractions we already know that the above-stated equivalence holds, if κ is a refinement of κ' (Prop. 4.16). This on the other hand can be traced back to the minimal models of φ according to the corresponding total preorders of κ and κ', since these are the interpretations that are added to the prior most plausible interpretations $[\![\kappa]\!]$ and $[\![\kappa']\!]$ (Th. 4.15). Thus, we know if κ refines κ', the prior most plausible interpretations are irrelevant for the fulfilment of **(DFPes-2)**$_\mathcal{L}$. Since the most plausible interpretations after revising κ and κ' exactly correspond to the minimal models of φ (Th. 3.45), we know that Prop. 4.16, which says that minimal change c-contractions satisfy **(DFPes-2)**$_\mathcal{L}$ in cases that κ refines κ', also holds for c-revisions, and even more general for all belief change operators satisfying **(AGMes$*$1)**-**(AGMes$*$6)** (Prop. 4.37).

Proposition 4.37. *Let κ and κ' be OCFs over the same signature Σ and $*$ a belief change operator satisfying **(AGMes$*$1)**-**(AGMes$*$6)**, then the following holds:*

$$\textit{If } \kappa \sqsubseteq \kappa', \ \textit{then} \ * \ \textit{satisfies } \textbf{(DFPes-2)}_\mathcal{L}.$$

Proof of (Prop. 4.37). In order to prove Prop. 4.37, we show that the conclusion of **(DFPes-2)**$_\mathcal{L}$, namely $Bel(\kappa * \varphi) \models Bel(\kappa' * \varphi)$, directly concludes from $\kappa \sqsubseteq \kappa'$. Thus, the antecedence of **(DFPes-2)**$_\mathcal{L}$ is irrelevant, given the assumption $\kappa \sqsubseteq \kappa'$.

$$\begin{aligned}
& Bel(\kappa * \varphi) \models Bel(\kappa' * \varphi) & \\
\Leftrightarrow\ & [\![\kappa * \varphi]\!] \subseteq [\![\kappa' * \varphi]\!] & \text{(Prop. 2.41)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} \subseteq \min\{[\![\varphi]\!], \preceq_{\kappa'}\} & \text{(Th. 3.46)} \\
\Leftarrow\ & \kappa \sqsubseteq \kappa' & \text{(Th. 4.15)}
\end{aligned}$$

□

Furthermore, we can also show that the refinement relation between κ and κ' can be retained, when choosing the revision parameters γ^+ and γ^- according to an appropriate revision strategy (Prop. 4.38). For more insights on strategic revisions we refer to the work of Sezgin et al. [SKIB20].

Proposition 4.38. *Let κ, κ' be OCFs over the same signature Σ and $\varphi \in \mathcal{L}_\Sigma$ a formulas. Then there exist parameters $\gamma_\kappa^-, \gamma_\kappa^+, \gamma_{\kappa'}^-, \gamma_{\kappa'}^+$ for c-revisions $\kappa \circledast \varphi$ and $\kappa' \circledast \varphi$ such that*

$$\textit{if } \kappa \sqsubseteq \kappa', \ \textit{then } \kappa \circledast \varphi \sqsubseteq \kappa' \circledast \varphi,$$

where $\gamma_\kappa^-, \gamma_\kappa^+, \gamma_{\kappa'}^-, \gamma_{\kappa'}^+$ originate from the definition of c-changes (Def. 3.35).

Proof of Prop. 4.38. Since the refinement stated in the consequence of Prop. 4.38 is equivalent to

$$\begin{aligned}
& \kappa \circledast \varphi \sqsubseteq \kappa' \circledast \varphi & \\
\Leftrightarrow\ & \text{if } \omega \preceq_{\kappa\circledast\varphi} \omega', \ \text{then } \omega \preceq_{\kappa'\circledast\varphi} \omega' & \text{(Def. 2.56)} \\
\Leftrightarrow\ & \text{if } \kappa \circledast \varphi\ (\omega) \leq \kappa \circledast \varphi\ (\omega'), \ \text{then } \kappa' \circledast \varphi\ (\omega) \leq \kappa' \circledast \varphi\ (\omega') & (4.5)
\end{aligned}$$

for all $\omega, \omega' \in \Omega_\Sigma$, we distinguish three cases in order to prove Prop. 4.38. In the first case, we assume both ω and ω' to satisfy φ or falsify φ, respectively. In the second case, we assume $\omega \models \varphi$ and $\omega' \not\models \varphi$, while we assume $\omega \not\models \varphi$ and $\omega' \models \varphi$ in the third case. We show that in all three cases the implication stated in Eq. 4.5 holds, if we chose the revision parameters according to the additional restriction

$$\gamma^- - \gamma^+ > \max\{\kappa(\omega) \mid \omega \models \varphi\} - \kappa(\neg\varphi) \tag{4.6}$$

for both κ and κ'. At this point, we briefly want to recall the general form of propositional c-revisions (Def. 3.44):

$$\kappa \circledast \varphi\ (\omega) = \kappa(\omega) - \kappa_0 + \begin{cases} \gamma^+, & \text{if } \omega \models \varphi \\ \gamma^-, & \text{if } \omega \models \neg\varphi \end{cases}$$

Case $\omega, \omega' \models \varphi$ or $\omega, \omega' \not\models \varphi$:
Since we assume both ω and ω' to satisfy φ or falsify φ, a further differentiation between γ^+ and γ^- for both OCFs is obsolete. Thus, we define the auxiliary variable

$$\gamma^{\pm} = \begin{cases} \gamma^+, & \omega \models \varphi \\ \gamma^-, & \omega \not\models \varphi \end{cases}.$$

$$\begin{aligned}
&\text{if } \kappa \circledast \varphi\ (\omega) \leq \kappa \circledast \varphi\ (\omega'),\ \text{then } \kappa' \circledast \varphi\ (\omega) \leq \kappa' \circledast \varphi\ (\omega') \\
\Leftrightarrow\ &\text{if } \kappa(\omega) - \kappa_0 + \gamma^{\pm}_{\kappa} \leq \kappa(\omega') - \kappa_0 + \gamma^{\pm}_{\kappa},\ \text{then } \kappa'(\omega) - \kappa'_0 + \gamma^{\pm}_{\kappa'} \leq \kappa'(\omega') - \kappa'_0 + \gamma^{\pm}_{\kappa'} \\
\Leftrightarrow\ &\text{if } \kappa(\omega) \leq \kappa(\omega'),\ \text{then } \kappa'(\omega) \leq \kappa'(\omega') \\
\Leftrightarrow\ &\kappa \sqsubseteq \kappa'
\end{aligned}$$

Due to the assumption that κ is a refinement of κ', we can conclude that the refinement property is retained after the revision for all pairs of interpretations that agree on the satisfiability of φ.

Case $\omega \models \varphi, \omega' \not\models \varphi$:

$$\begin{aligned}
&\text{if } \kappa \circledast \varphi\ (\omega) \leq \kappa \circledast \varphi\ (\omega'),\ \text{then } \kappa' \circledast \varphi\ (\omega) \leq \kappa' \circledast \varphi\ (\omega') \\
\Leftrightarrow\ &\text{if } \kappa(\omega) - \kappa_0 + \gamma^{+}_{\kappa} \leq \kappa(\omega') - \kappa_0 + \gamma^{-}_{\kappa},\ \text{then } \kappa'(\omega) - \kappa'_0 + \gamma^{+}_{\kappa'} \leq \kappa'(\omega') - \kappa'_0 + \gamma^{-}_{\kappa'} \\
\Leftrightarrow\ &\text{if } \kappa(\omega) + \gamma^{+}_{\kappa} \leq \kappa(\omega') + \gamma^{-}_{\kappa},\ \text{then } \kappa'(\omega) + \gamma^{+}_{\kappa'} \leq \kappa'(\omega') + \gamma^{-}_{\kappa'} \\
\Leftrightarrow\ &\text{if } \kappa(\omega) - \kappa(\omega') \leq \gamma^{-}_{\kappa} - \gamma^{+}_{\kappa},\ \text{then } \kappa'(\omega) - \kappa'(\omega') \leq \gamma^{-}_{\kappa'} - \gamma^{+}_{\kappa'} \\
\Leftarrow\ &\text{if } \kappa(\omega) - \kappa(\omega') \leq \max\{\kappa(\omega) \mid \omega \models \varphi\} - \kappa(\neg\varphi), \\
&\text{then } \kappa'(\omega) - \kappa'(\omega') \leq \max\{\kappa'(\omega) \mid \omega \models \varphi\} - \kappa'(\neg\varphi) \\
\Leftrightarrow\ &\text{if } \underbrace{\kappa(\omega) - \max\{\kappa(\omega) \mid \omega \models \varphi\}}_{\leq 0} \leq \underbrace{\kappa(\neg\varphi) - \kappa(\omega')}_{\geq 0}, \\
&\text{then } \underbrace{\kappa'(\omega) - \max\{\kappa'(\omega) \mid \omega \models \varphi\}}_{\leq 0} \leq \underbrace{\kappa'(\neg\varphi) - \kappa'(\omega')}_{\geq 0}
\end{aligned}$$

Since $\kappa'(\omega)$ must be smaller than or equal to the rank of the most implausible model of φ, and $\kappa'(\neg\varphi)$ must be smaller than or equal to the rank of any model of $\neg\varphi$, we know that $\kappa'(\omega) - \max\{\kappa'(\omega) \mid \omega \models \varphi\}$ can at most be equal to $\kappa'(\neg\varphi) - \kappa'(\omega')$. The same holds for κ'. Therefore, we can conclude that the refinement property is also retained for pairs of interpretations ω, ω' with $\omega \models \varphi$ and $\omega \not\models \varphi$.

Case $\omega \not\models \varphi, \omega' \models \varphi$:
For the case that ω falsifies φ, while ω' satisfies φ, we prove that the refinement is retained by showing that the antecedence of Eq. 4.5 $\kappa \circledast \varphi\ (\omega) \leq \kappa \circledast \varphi\ (\omega')$ does not hold in the first place, and therefore the implication is fulfilled, since a relation that does not hold in $\preceq_{\kappa\circledast\varphi}$ does not necessarily have to hold in $\preceq_{\kappa'\circledast\varphi}$.

$$
\begin{aligned}
& \kappa \circledast \varphi\ (\omega) \leq \kappa \circledast \varphi\ (\omega') \\
\Leftrightarrow\ & \kappa(\omega) - \kappa_0 + \gamma_\kappa^- \leq \kappa(\omega') - \kappa_0 + \gamma_\kappa^+ \\
\Leftrightarrow\ & \kappa(\omega) + \gamma_\kappa^- \leq \kappa(\omega') + \gamma_\kappa^+ \\
\Leftrightarrow\ & \gamma_\kappa^- - \gamma_\kappa^+ \leq \kappa(\omega') - \kappa(\omega) \\
\Rightarrow\ & \max\{\kappa(\omega) \mid \omega \models \varphi\} - \kappa(\neg\varphi) < \kappa(\omega') - \kappa(\omega) \\
\Leftrightarrow\ & \underbrace{\kappa(\omega) - \kappa(\neg\varphi)}_{\geq 0} < \underbrace{\kappa(\omega') - \max\{\kappa(\omega) \mid \omega \models \varphi\}}_{\leq 0}
\end{aligned}
$$

Since $\kappa(\omega) - \kappa(\neg\varphi)$ is always greater or equal to $\kappa(\omega') - \max\{\kappa(\omega) \mid \omega \models \varphi\}$, we know that $\gamma_\kappa^- - \gamma_\kappa^+ \leq \kappa(\omega') - \kappa(\omega)$ cannot hold either. Thus, we know that if $\omega' \models \varphi$ and $\omega \not\models \varphi$, then $\kappa \circledast \varphi\ (\omega)$ must be greater than $\kappa \circledast \varphi\ (\omega')$. Since the antecedence does not hold in this case, we know that the implication stated in Eq. 4.5 holds.

In conclusion, we showed in the three cases above that there exist parameters $\gamma_\kappa^-, \gamma_\kappa^+, \gamma_{\kappa'}^-, \gamma_{\kappa'}^+$ for c-revisions $\kappa \circledast \varphi$ and $\kappa' \circledast \varphi$, and all $\omega, \omega' \in \Omega_\Sigma$ such that the refinement property $\kappa \sqsubseteq \kappa'$ is retained after the revision. □

In the proof of Prop. 4.38, we showed that the refinement relation of two OCFs can be preserved after revising them with a certain formula, when choosing the revision properties according to an appropriate strategy. We proved the existence of such a strategy by giving a concrete example (see Eq. 4.6). However, other strategies might retain the refinement relation as well. In the following, we like to illustrate Prop. 4.38 and show that choosing the parameters according to Eq. 4.6 always retains the refinement relation (Ex. 4.9). In addition to this, we want to compare the above-mentioned strategy to a more simplistic one, which chooses the same parameters for both OCFs, and show that it is inappropriate for retaining the refinement relation.

Example 4.9. *This example illustrates the maintenance of the refinement of two OCFs $\kappa \sqsubseteq \kappa'$ when revising them with the same formula φ as stated in Prop. 4.38, and further illustrates that it is not sufficient to just choose the same parameters for both OCFs. For this we assume κ and κ' as in Tab. 32 and $\varphi \equiv p$.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	pbf, $p\overline{b}\,\overline{f}$
2	$p\overline{b}f$, $\overline{p}bf$
1	$\overline{p}\overline{b}f$
0	$pb\overline{f}$, $\overline{p}b\overline{f}$, $\overline{p}\overline{b}\,\overline{f}$

$\kappa'(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	-
2	$p\overline{b}f$, $\overline{p}bf$, pbf, $p\overline{b}\,\overline{f}$
1	-
0	$pb\overline{f}$, $\overline{p}b\overline{f}$, $\overline{p}\overline{b}\,\overline{f}$, $\overline{p}\overline{b}f$

Table 32: OCFs κ and κ' over signature Σ_{Tweety}, where $\kappa \sqsubseteq \kappa'$.

Furthermore, we choose the revision parameters according to the same additional restriction (Eq. 4.5) as used in the proof of Prop. 4.38. Choosing the revision parameters γ^+, γ^- this way obviously is on par with the parameter restrictions given by the definition of propositional c-revisions (Def. 3.44):

$$\gamma^- - \gamma^+ > \max\{\kappa(\omega) \mid \omega \models \varphi\} - \kappa(\neg\varphi) > \kappa(\varphi) - \kappa(\neg\varphi)$$

For the revisions $\kappa \circledast \varphi$ and $\kappa' \circledast \varphi$ this means that the parameters must fulfil

$$\begin{aligned} \gamma^-_\kappa - \gamma^+_\kappa &> 3 - 0 = 3, \\ \gamma^-_{\kappa'} - \gamma^+_{\kappa'} &> 2 - 0 = 2. \end{aligned}$$

Therefore, we choose $\gamma^-_{\kappa'} = 4$, $\gamma^+_{\kappa'} = 0$, $\gamma^-_\kappa = 3$ and $\gamma^+_\kappa = 0$ without loss of generality. Revising κ and κ' with φ then results in the OCFs stated in Tab. 33.

$\kappa \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
6	$\overline{p}bf$
5	$\overline{p}\overline{b}f$
4	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\,\overline{f}$
3	pbf, $p\overline{b}\,\overline{f}$
2	$p\overline{b}f$
1	-
0	$pb\overline{f}$

$\kappa' \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
6	-
5	$\overline{p}bf$
4	-
3	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\,\overline{f}$, $\overline{p}\overline{b}f$
2	$p\overline{b}f$, pbf, $p\overline{b}\,\overline{f}$
1	-
0	$pb\overline{f}$

Table 33: Revisions of κ and κ' (Tab. 32) with $\varphi \equiv p$ and parameters $\gamma^- = 4$, $\gamma^+ = 0$, $\kappa_0 = 0$ for $\kappa \circledast \varphi$, and $\gamma^- = 3$, $\gamma^+ = 0$, $\kappa_0 = 0$ for $\kappa' \circledast \varphi$.

As shown in Prop. 4.38, we see that $\kappa \circledast \varphi$ refines $\kappa' \circledast \varphi$, and therefore the refinement relation of the prior OCFs is retained. Choosing the parameters as described *above guarantees the maintenance of the refinement relation, because the rank of* $\neg\varphi$ *is greater than the rank of any model of* φ*. Thus, the models of* φ *and* $\neg\varphi$ *become separated such that there exists a rank* r *with* $\{\omega \in \Omega \mid \kappa \circledast \varphi\ (\omega) < r\} = [\![\varphi]\!]$ *and* $\{\omega \in \Omega \mid \kappa \circledast \varphi\ (\omega) \geq r\} = [\![\neg\varphi]\!]$*. In the revised OCFs above* r *equals* γ_{κ}^{-} *and* $\gamma_{\kappa'}^{-}$*, respectively. Moreover, we know that due to the principle of conditional preserving the order of the models of* φ *remains unchanged, and so does the order of the models of* $\neg\varphi$*. Therefore,* $\kappa \circledast \varphi$ *must refine* $\kappa' \circledast \varphi$*, if* $\kappa \sqsubseteq \kappa'$*.*

This especially illustrates that it is not possible to just reuse the parameters from $\kappa \circledast \varphi$ *for* $\kappa' \circledast \varphi$ *or vice-versa, since without further assumptions it is possible that the underlying total preorders change in a manner that the refinement relation of the prior OCFs cannot be retained, even though the parameters might be valid according to the restriction given in the definition of c-revisions. We show this by giving a counter example, in which we again assume* κ *and* κ' *as in Tab. 32. For the revision parameters we choose* $\gamma^{-} = 1$*,* $\gamma^{+} = 0$ *and* $\kappa_0 = 0$ *for both* κ *and* κ'*, which result in* $\kappa \circledast \varphi$ *and* $\kappa' \circledast \varphi$ *as given in Tab. 34 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{p}bf$
2	$p\overline{b}f$, pbf, $p\overline{b}\overline{f}$
1	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$, $\overline{p}\overline{b}f$
0	$pb\overline{f}$

$\kappa'(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
4	-
3	$\overline{p}bf$, pbf, $p\overline{b}\overline{f}$
2	$\overline{p}\overline{b}f$, $p\overline{b}f$
1	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$
0	$pb\overline{f}$

Table 34: Revisions $\kappa \circledast \varphi$, $\kappa' \circledast \varphi$, with $\varphi \equiv p$, $\gamma^{-} = 1$, $\gamma^{+} = 0$, $\kappa_0 = 0$ for both revisions, and OCFs κ and κ' with $\kappa' \sqsubseteq \kappa$ as given in Tab. 21.

Since the orders of the interpretations do not have to be equal in both total preorders $\preceq_{\kappa}$ *and* $\preceq_{\kappa'}$*, and the ranks of the models of* φ *and* $\neg\varphi$ *are shifted by different values (here* 1 *and* 0*), it is possible to induce changes in* $\preceq_{\kappa'}$ *that are not induced in* $\preceq_{\kappa}$*. Given the posterior OCFs in Tab. 34, we see for example that* $\mathrm{p\overline{b}f} \preceq_{\kappa'} \mathrm{\overline{p}\overline{b}f}$ *holds, but not* $\mathrm{p\overline{b}f} \preceq_{\kappa} \overline{p}\overline{b}f$*. Therefore, the revisions changed the prior OCFs, such that the refinement property is not maintained.*

Next, we examine **(DFPes-3)**$_{\mathcal{L}}$ for c-revisions. In Section 4.2, we showed that minimal change c-contractions do not satisfy **(DFPes-3)**$_{\mathcal{L}}$, and furthermore that neither the $\models$ nor the $=\!\!|$ direction of the there stated equivalence holds generally, since the minimal models that were added to the prior most plausible interpretations are potentially disjunct. In contrast to minimal change c-contractions, it can be shown that propositional c-revisions are capable of satisfying **(DFPes-3)**$_{\mathcal{L}}$ (Prop. 4.39).

Proposition 4.39. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circledast$ a propositional c-revision, then $\circledast$ satisfies*

$$\text{if } \varphi \models \psi, \text{ then } Bel(\Psi \circledast \varphi) \equiv Bel((\Psi \circledast \psi) \circledast \varphi). \qquad \textbf{(DFPes-3)}_{\mathcal{L}}$$

Prop. 4.39 concludes directly from the fact that $\textbf{(DFPes-3)}_{\mathcal{L}}$ is equivalent to **(DP1)** (see Section 2.3 or Appendix A.1), for which we already know that propositional c-revision are capable of satisfying of (Prop. 3.48). In order to understand why the subsequently performed expansion changes the beliefs in a way that the third forgetting postulate $\textbf{(DFPes-3)}_{\mathcal{L}}$ is satisfied for c-revision, but not for c-contractions, we briefly recap why minimal change c-contractions are not capable of satisfying $\textbf{(DFPes-3)}_{\mathcal{L}}$. Due to the definition of minimal change c-contractions (Def. 3.37 and Prop. 3.41), we know that the posterior most plausible interpretations consist of the prior most plausible interpretations and the minimal models that contradict the formula we like to forget. When we apply the minimal change c-contraction as stated in $\textbf{(DFPes-3)}_{\mathcal{L}}$, the following equation must hold in order to satisfy this postulate:

$$\begin{aligned} & Bel(\kappa \ominus \varphi) \equiv Bel((\kappa \ominus \psi) \ominus \varphi) \\ \Leftrightarrow\ & [\![\kappa]\!] \cup \min\{[\![\neg\varphi]\!], \preceq_\kappa\} = [\![\kappa]\!] \cup \min\{[\![\neg\psi]\!], \preceq_\kappa\} \cup \min\{[\![\neg\varphi]\!], \preceq_{\kappa\ominus\psi}\}. \end{aligned} \tag{4.7}$$

Since $\textbf{(DFPes-3)}_{\mathcal{L}}$ assumes $\varphi \models \psi$, we can conclude that the above-mentioned Eq. 4.7 cannot hold in general, because adding the minimal models of $\neg\varphi$ to rank 0 will not have any influence on $[\![\kappa]\!]$ after the minimal models of $\neg\psi$ were added to rank 0, since they are models of $\neg\varphi$ as well. However, this way contracting φ and contracting ψ and φ consecutively will not result in equivalent posterior beliefs. The reason why this changes for the revision lies in the relation of the formulas φ and ψ. As already mentioned above, $\textbf{(DFPes-3)}_{\mathcal{L}}$ assumes ψ to be inferable from φ. When we apply the c-revision as stated in $\textbf{(DFPes-3)}_{\mathcal{L}}$, we know that

$$\begin{aligned} & ([\![\kappa]\!] \setminus [\![\neg\varphi]\!]) \cup \min\{[\![\varphi]\!], \preceq_\kappa\} \\ = & ([\![\kappa]\!] \setminus ([\![\neg\psi]\!] \cup [\![\neg\varphi]\!])) \cup (\min\{[\![\psi]\!], \preceq_\kappa\} \setminus [\![\neg\varphi]\!]) \cup \min\{[\![\varphi]\!], \preceq_{\kappa\ominus\neg\psi}\} \end{aligned} \tag{4.8}$$

must hold in order to satisfy $\textbf{(DFPes-3)}_{\mathcal{L}}$. Beside the set differences, the most important difference between Eq. 4.7 and Eq. 4.8 is that in case of the revision we do not forget φ and ψ, but $\neg\varphi$ and $\neg\psi$. Thus, the relation of the formulas we like to forget is inverted. This affects the minimal models added to rank 0. When first forgetting $\neg\psi$, i.e. adding $\min\{[\![\psi]\!], \preceq_\kappa\}$ to the most plausible interpretations, it is possible to either add some models of φ to rank 0 as well or only add those models of ψ that do not satisfy φ. This is different to the contraction for which we know that there will definitely be models of $\neg\varphi$ added to rank 0 after forgetting ψ. When forgetting $\neg\varphi$ afterwards, there are two possible cases. Either the previous forgetting of $\neg\psi$ already added some models of φ to rank 0, or none of the models of φ were added to rank 0 so far. In the first case we know that if the minimal models of ψ contain models of φ, then they must be the minimal models of φ, since $\varphi \models \psi$: If $\varphi \models \psi$ and $\min\{[\![\psi]\!], \preceq_\kappa\} \cap [\![\varphi]\!] \neq \emptyset$, then

$$([\![\kappa]\!] \setminus ([\![\neg\psi]\!] \cup [\![\neg\varphi]\!])) \cup (\min\{[\![\psi]\!], \preceq_\kappa\} \setminus [\![\neg\varphi]\!]) \cup \min\{[\![\varphi]\!], \preceq_{\kappa\ominus\neg\psi}\}$$

$$= (\llbracket\kappa\rrbracket \setminus (\llbracket\neg\psi\rrbracket \cup \llbracket\neg\varphi\rrbracket)) \cup \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\}.$$

In the other case, the minimal models of φ are just added to rank 0, because the minimal models of φ are not affected by the previous revision due to the principle of conditional preservation: If $\varphi \models \psi$ and $\min\{\llbracket\psi\rrbracket, \preceq_\kappa\} \cap \llbracket\varphi\rrbracket = \emptyset$, then

$$\begin{aligned} &(\llbracket\kappa\rrbracket \setminus (\llbracket\neg\psi\rrbracket \cup \llbracket\neg\varphi\rrbracket)) \cup (\min\{\llbracket\psi\rrbracket, \preceq_\kappa\} \setminus \llbracket\neg\varphi\rrbracket) \cup \min\{\llbracket\varphi\rrbracket, \preceq_{\kappa\ominus\neg\psi}\} \\ =\ &(\llbracket\kappa\rrbracket \setminus (\llbracket\neg\psi\rrbracket \cup \llbracket\neg\varphi\rrbracket)) \cup \min\{\llbracket\varphi\rrbracket, \preceq_\kappa\}. \end{aligned}$$

Either way we can conclude that the minimal models of φ are assigned to rank 0 after performing both revisions subsequently. Finally, the models contradicting φ that were added by the contraction of $\neg\psi$ are removed from the most plausible interpretations by means of the expansion. Therefore, we know that the most plausible interpretations after revising with φ or ψ and φ subsequently are the same.

In the following example Ex. 4.10, we want to illustrate why the way c-revisions affect the most plausible interpretations ensures that **(DFPes-3)$_\mathcal{L}$** is satisfied as discussed above.

Example 4.10. *This example illustrates **(DFPes-3)$_\mathcal{L}$** for c-revisions by showing how they affect the most plausible interpretations. For this we consider the OCF κ given in Tab. 35 below as the prior epistemic state. On this OCF we will perform several c-revisions. For the formulas we like to revise with, we assume $\varphi \equiv p$ and $\psi \equiv p \vee f$, which fulfil the condition $\varphi \models \psi$ assumed by **(DFPes-3)$_\mathcal{L}$**. The OCF $\kappa \circledast \varphi$ (Tab. 35) shows that due to revising with p the minimal models of p are added to $\llbracket\kappa\rrbracket$ and at the same time all models of $\neg p$ are shifted such that none of them is assigned to rank 0 anymore. The removal of the models of $\neg p$ from $\llbracket\kappa\rrbracket$ guarantees that p is believed by $\kappa \circledast \varphi$.*

*According to **(DFPes-3)$_\mathcal{L}$**, we obtain the same most plausible interpretations when first revising with a formula ψ, that can be inferred by φ, and then revising with φ afterwards. For this we consider the OCF $\kappa \circledast \psi$ (Tab. 35) that results after revising κ with $p \vee f$. Since all most plausible interpretations of $\llbracket\kappa\rrbracket$ falsify $p \vee f$, they are removed from rank 0, while the minimal models of $p \vee f$ are added.*

Next, we revise $\kappa \circledast \psi$ with p and obtain $\kappa_r^\circ \circledast \varphi$ (Tab. 35), where κ_r° represents the result of the previous revision. Since the minimal models of $p \vee f$ in κ also consists of models of p, we know that the minimal models of p are assigned to rank 0 after the revision $\kappa \circledast \psi$. Thus, no new interpretations must be added to rank 0. Anyhow, due to the previous revision with $p \vee f$ there still exist interpretations with rank 0 that satisfy $p \vee f$, but contradict p. These interpretations are removed by the revision $\kappa_r^\circ \circledast \varphi$, such that only the minimal models of p remain.

In conclusion, the most plausible interpretations after the revisions $\kappa \circledast \varphi$ and $\kappa_r^\circ \circledast \varphi$ are the same and so their beliefs are equivalent. Nonetheless, the OCFs are not equal themselves, since the ranks they assign to the remaining interpretations can differ.

Next, we examine the fourth forgetting postulate **(DFPes-4)$_\mathcal{L}$** for c-revisions. In Section 4.2 we already showed that minimal change c-contractions are neither capable of satisfying **(DFPes-4)$_\mathcal{L}$**, nor its $\models$ or $\dashv$ direction, since this would

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	$pb\overline{f}$, $p\overline{b}f$
1	$\overline{p}bf$, pbf, $p\overline{b}\overline{f}$, $\overline{p}\overline{b}f$
0	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$

$\kappa \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	$\overline{p}bf$, $\overline{p}\overline{b}f$
1	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$, $pb\overline{f}$, $p\overline{b}f$
0	pbf, $p\overline{b}\overline{f}$

$\kappa \circledast \psi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	-
1	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$, $pb\overline{f}$, $p\overline{b}f$
0	$\overline{p}bf$, pbf, $p\overline{b}\overline{f}$, $\overline{p}\overline{b}f$

$\kappa_r^\circ \circledast \varphi\ (\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
3	-
2	$\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$
1	$\overline{p}bf$, $\overline{p}\overline{b}f$, $pb\overline{f}$, $p\overline{b}f$
0	pbf, $p\overline{b}\overline{f}$

Table 35: Revisions of $\psi \equiv p \vee f$ and $\varphi \equiv p$. $\gamma^+ = 0$ for all of the stated revisions. Top left: OCF κ over signature Σ_{Tweety}. Top right: Result of revising κ with p and parameters $\kappa_0 = 1, \gamma^- = 2$. Bottom left: Result of revising κ with $p \vee f$ and parameters $\kappa_0 = 1, \gamma^- = 2$. Bottom Right: Result of revising $\kappa_r^\circ = \kappa \circledast \psi$ with p and parameters $\kappa_0 = 0, \gamma^- = 1$.

require the minimal models of $\neg\varphi \wedge \neg\psi$ to be equal to the unification of the minimal models of φ and ψ. For c-revisions we will show in the following that they are not capable of generally satisfying **(DFPes-4)$_\mathcal{L}$** either. However, other than for minimal change c-contractions, we are able to show that **(DFPes-4)$_\mathcal{L}$** holds under certain assumptions on the formulas φ and ψ we revise the OCF κ with, and that the $\models$ direction of **(DFPes-4)$_\mathcal{L}$** even holds in general.

First, we want to argue that **(DFPes-4)$_\mathcal{L}$** cannot be satisfied by c-revisions in general, since this would require the minimal models of $\neg\varphi \vee \neg\psi$ to be equal to the unification of the minimal models of φ and ψ, which is similar to the requirement for minimal change c-contractions. However, the fact that we consider the minimal models of $\neg\varphi \vee \neg\psi$ instead of $\neg\varphi \wedge \neg\psi$ in this case allows us to formulate further assumptions on φ and ψ, such that the equivalence stated in **(DFPes-4)$_\mathcal{L}$** holds when applying the c-revision to those formulas (Prop. 4.40).

Proposition 4.40. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circledast$ a propositional c-revision, then*

$$Bel(\kappa \circledast \varphi \vee \psi) \equiv Bel(\kappa \circledast \varphi) \cap Bel(\kappa \circledast \psi), \qquad \textbf{(DFPes-4)}_\mathcal{L}$$

if and only if $\kappa(\varphi) = \kappa(\psi)$.

Proof of (Prop. 4.40).

$$
\begin{aligned}
& Bel(\kappa \circledast \varphi \vee \psi) \equiv Bel(\kappa \circledast \varphi) \cap Bel(\kappa \circledast \psi) & \\
\Leftrightarrow\ & Th(\llbracket \kappa \circledast \varphi \vee \psi \rrbracket) \equiv Th(\llbracket \kappa \circledast \varphi \rrbracket) \cap Th(\llbracket \kappa \circledast \psi \rrbracket) & \text{(Prop. 2.38)} \\
\Leftrightarrow\ & Th(\llbracket \kappa \circledast \varphi \vee \psi \rrbracket) \equiv Th(\llbracket \kappa \circledast \varphi \rrbracket \cup \llbracket \kappa \circledast \psi \rrbracket) & \text{(Lem. 2.25)} \\
\Leftrightarrow\ & \llbracket \kappa \circledast \varphi \vee \psi \rrbracket = \llbracket \kappa \circledast \varphi \rrbracket \cup \llbracket \kappa \circledast \psi \rrbracket & \text{(Prop. 2.38)} \\
\Leftrightarrow\ & \min\{\llbracket \varphi \vee \psi \rrbracket, \preceq_\kappa\} = \min\{\llbracket \varphi \rrbracket, \preceq_\kappa\} \cup \min\{\llbracket \varphi \rrbracket, \preceq_\kappa\} & \text{(Th. 3.45)} \\
\Leftrightarrow\ & \kappa(\varphi) = \kappa(\psi) & \text{(Prop. 2.52)}
\end{aligned}
$$

□

As shown in Prop. 4.40 above, c-revisions are capable of satisfying the equivalence stated in **(DFPes-4)$_\mathcal{L}$**, if and only if the formulas we revise κ with are equally plausible. From Th. 3.45, we know that the most plausible interpretations after the revision correspond to the minimal models of the formula we revised our prior epistemic state with. Thus, $\min\{\llbracket \varphi \vee \psi \rrbracket, \preceq_\kappa\} = \min\{\llbracket \varphi \rrbracket, \preceq_\kappa\} \cup \min\{\llbracket \varphi \rrbracket, \preceq_\kappa\}$ must hold in order to satisfy **(DFPes-4)$_\mathcal{L}$**. Furthermore, we know from Prop. 2.52 that this equation holds, if and only if $\kappa(\varphi) = \kappa(\psi)$. Therefore, we know in conclusion that **(DFPes-4)$_\mathcal{L}$** holds, if and only if φ and ψ are equally plausible. However, if one of the formulas is more plausible than the other, we know that the minimal models of $\varphi \vee \psi$ correspond to the minimal models of the more plausible formula φ or ψ (Lem. 2.53). This allows us to further conclude that the $\models$ direction of **(DFPes-4)$_\mathcal{L}$** must even hold in general (Prop. 4.41).

Proposition 4.41. *Let κ be an OCF, $\varphi, \psi \in \mathcal{L}$ formulas and $\circledast$ a propositional c-revision, then the following holds:*

$$Bel(\kappa \circledast \psi \vee \varphi) \models Bel(\kappa \circledast \psi) \cap Bel(\kappa \circledast \varphi)$$

Proof of Prop. 4.41.

$$
\begin{aligned}
& Bel(\kappa \circledast \varphi \vee \psi) \models Bel(\kappa \circledast \varphi) \cap Bel(\kappa \circledast \psi) & \\
\Leftrightarrow\ & Th(\llbracket \kappa \circledast \varphi \vee \psi \rrbracket) \models Th(\llbracket \kappa \circledast \varphi \rrbracket) \cap Th(\llbracket \kappa \circledast \psi \rrbracket) & \text{(Prop. 2.41)} \\
\Leftrightarrow\ & Th(\llbracket \kappa \circledast \varphi \vee \psi \rrbracket) \models Th(\llbracket \kappa \circledast \varphi \rrbracket \cup \llbracket \kappa \circledast \psi \rrbracket) & \text{(Lem. 2.25)} \\
\Leftrightarrow\ & \llbracket \kappa \circledast \varphi \vee \psi \rrbracket \subseteq \llbracket \kappa \circledast \varphi \rrbracket \cup \llbracket \kappa \circledast \psi \rrbracket & \text{(Prop. 2.41)} \\
\Leftrightarrow\ & \min\{\llbracket \varphi \vee \psi \rrbracket, \preceq_\kappa\} \subseteq \min\{\llbracket \varphi \rrbracket, \preceq_\kappa\} \cup \min\{\llbracket \psi \rrbracket, \preceq_\kappa\} & \text{(Th. 3.45)}
\end{aligned}
$$

From Lem. 2.53, we can conclude that the above-stated subset relation holds, since $\min\{\llbracket \varphi \vee \psi \rrbracket, \preceq_\kappa\}$ must either be the minimal models of φ or ψ, or must be equal to their unification. Thus, we further know that Prop. 4.41 is generally satisfied by c-revisions. □

Lastly, we examine **(DFPes-5)$_\mathcal{L}$** for c-revisions. For minimal change c-contractions we were able to show that neither **(DFPes-5)$_\mathcal{L}$** nor one of the directions of the stated equivalence ($\models$ or $\dashv$) holds. Furthermore, we were able to show that $\textbf{(DFPes-5)}_{\mathcal{L}}$ holds for those minimal change c-contractions applied to formulas φ and ψ with $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\psi]\!], \preceq_\kappa\} \neq \emptyset$ (Th. 4.26). In the following, we show that $\textbf{(DFPes-5)}_{\mathcal{L}}$ cannot be generally satisfied by c-revisions either, and that the further restriction under which the postulate holds for minimal change c-contractions is not sufficient for c-revisions to satisfy it. Instead, we show that $\textbf{(DFPes-5)}_{\mathcal{L}}$ is satisfied by c-revisions when applied to formulas φ and ψ with $\min\{[\![\varphi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_\kappa\}$.

First, we want to argue why the assumption $\min\{[\![\varphi]\!], \preceq_\kappa\} \cap \min\{[\![\varphi]\!], \preceq_\kappa\} \neq \emptyset$ is not sufficient for c-revisions to satisfy $\textbf{(DFPes-5)}_{\mathcal{L}}$. Considering the posterior belief sets as stated in $\textbf{(DFPes-5)}_{\mathcal{L}}$, we know that the following equivalence must hold.

$$
\begin{aligned}
& Bel(\kappa \circledast \varphi \vee \psi) \equiv Bel((\kappa \circledast \varphi) \circledast \psi) \\
\Leftrightarrow\ & [\![\kappa \circledast \varphi \vee \psi]\!] = [\![(\kappa \circledast \varphi) \circledast \psi]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\ & \min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\} && \text{(Th. 3.45)}
\end{aligned}
$$

First of all, it can clearly be seen, that the equality of $\min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\}$ and $\min\{[\![\psi]\!], \preceq_{\kappa\circledast\varphi}\}$ cannot be guaranteed without further assumptions on φ and ψ. However, if we assume that the intersection of the minimal models of φ and ψ is not empty, we can admittedly conclude

$$
\begin{aligned}
& \min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\} && \text{(Lem. 2.51, Prop. 2.52)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\} = [\![\kappa \circledast \varphi]\!] \setminus [\![\neg\psi]\!] \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} \cup \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\} \setminus [\![\neg\psi]\!], && \text{(Th. 3.45)}
\end{aligned}
$$

but on the other hand this concludes that the assumption is not sufficient, since the above-stated equation also requires further assumptions on φ and ψ. Thus, the restrictions for minimal change c-contractions cannot be transferred to c-revisions.

Next, we want to show in Ex. 4.11 that $\textbf{(DFPes-5)}_{\mathcal{L}}$ does not hold for c-revisions in general. This can be traced back to the relation of the minimal models of $\varphi \vee \psi$ and the posterior minimal models of ψ after revising with φ, as already stated above.

Example 4.11. *This example illustrates that c-revisions does neither satisfy* $\textbf{\textit{(DFPes-5)}}_{\mathcal{L}}$ *nor one of the directions of the stated equivalence ($\models$ or $\models$) in general. For this we assume the OCF κ as in Tab. 36 below.*

$\kappa(\omega)$	$\omega \in \Omega_{\Sigma_{Tweety}}$
∞	-
$\vdots$	-
2	-
1	pbf, $p\overline{b}f$, $\overline{p}bf$, $\overline{p}\overline{b}f$
0	$pb\overline{f}$, $p\overline{b}\overline{f}$, $\overline{p}b\overline{f}$, $\overline{p}\overline{b}\overline{f}$

Table 36: OCF κ over signature Σ_{Tweety}.

For the formulas we want to revise κ with, we consider $\varphi \equiv p$ and $\psi \equiv f$. According to Th. 3.45, the posterior most plausible interpretations after revising κ with $p \vee f$ are

$$[\![\kappa \circledast p \vee f]\!] = \min\{[\![p \vee f]\!], \preceq_\kappa\} = \{pb\overline{f}, p\overline{b}\overline{f}\}.$$

When consecutively revising κ with p and f we obtain

$$[\![(\kappa \circledast p) \circledast f]\!] = \min\{[\![f]\!], \preceq_{\kappa \circledast p}\}.$$

The posterior most plausible interpretations after revising with p and f consecutively are not unique, but depend on the choice of the parameters γ^- and γ^+, which originate form the definition of c-changes (Def. 3.35). However, since $[\![\kappa \circledast p \vee f]\!]$ does not consist of any models of f, we know that

$$[\![(\kappa \circledast p) \circledast f]\!] \cap [\![\kappa \circledast p \vee f]\!] = \emptyset.$$

Therefore, we know that neither of the resulting beliefs can be inferred from the other.

As Ex. 4.11 illustrated, it is not possible for c-revisions to satisfy **(DFPes-5)**$_\mathcal{L}$ without further assumptions on the revised formulas φ and ψ. However, it seems obvious that the property stated in **(DFPes-5)**$_\mathcal{L}$ holds if the revision of $\varphi \vee \psi$ and φ result in equivalent beliefs such that the subsequent revision of ψ has no influence on the prior beliefs. This is the case if the prior minimal models of φ and ψ are equal. We show in Prop. 4.42 that this equality implies the fulfilment of **(DFPes-5)**$_\mathcal{L}$.

Proposition 4.42. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circledast$ a propositional c-revision, then the following holds:*

$$\textit{If } \min\{[\![\psi]\!], \preceq_\kappa\} = \min\{[\![\varphi]\!], \preceq_\kappa\}, \textit{ then } Bel(\kappa \circledast \varphi \vee \psi) \equiv Bel((\kappa \circledast \varphi) \circledast \psi).$$

Proof of Prop. 4.42. In the following, we prove the correctness of Prop. 4.42 above. For this we assume the equality of $\min\{[\![\psi]\!], \preceq_\kappa\}$ and $\min\{[\![\varphi]\!], \preceq_\kappa\}$ and further refer to it as $\varphi \overset{\min,\preceq_\kappa}{=} \psi$.

$$\begin{aligned}
& Bel(\kappa \circledast \varphi \vee \psi) \equiv Bel((\kappa \circledast \varphi) \circledast \psi) && \\
\Leftrightarrow\ & [\![\kappa \circledast \varphi \vee \psi]\!] = [\![(\kappa \circledast \varphi) \circledast \psi]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\ & \min\{[\![\varphi \vee \psi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\} && \text{(Th. 3.45)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_{\kappa \circledast \varphi}\} && \text{(Lem. 2.53)} \\
\Leftrightarrow\ & \min\{[\![\varphi]\!], \preceq_\kappa\} = \min\{[\![\psi]\!], \preceq_\kappa\} && (\varphi \overset{\min,\preceq_\kappa}{=} \psi, \text{ Prop. 3.53})
\end{aligned}$$

□

Thus, we showed that neither c-contractions nor c-revisions satisfy **(DFPes-5)**$_\mathcal{L}$ in general, and that the additional assumptions sufficient for minimal change c-contractions to satisfy **(DFPes-5)**$_\mathcal{L}$ cannot be transferred to c-revisions. In this case even stricter assumptions are necessary.

In Th. 4.43 and 4.44, we summarize the results we elaborated for revisions satisfying **(AGMes$*$1)**-**(AGMes$*$6)** and propositional c-revisions so far.

Theorem 4.43. *Let Ψ be an epistemic state equipped with a faithfully assigned total preorder $\preceq_\Psi$, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $*$ a belief change operator satisfying* ***(AGMes∗1)****-****(AGMes∗6)****, then the following holds:*

$*$ *satisfies* ***(DFPes-1)***$_\mathcal{L}$*, only if* $Bel(\Psi) \models \varphi$
$*$ *satisfies* ***(DFPes-3)***$_\mathcal{L}$
$*$ *falsifies* ***(DFPes-6)***$_\mathcal{L}$

Theorem 4.44. *Let κ be an OCF over signature Σ, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circledast$ a propositional c-revision, then the following holds:*

$\circledast$ *satisfies* ***(DFPes-1)***$_\mathcal{L}$*, only if* $Bel(\Psi) \models \varphi$
$\circledast$ *satisfies* ***(DFPes-2)***$_\mathcal{L}$*, if* $\kappa \sqsubseteq \kappa'$
$\circledast$ *satisfies* ***(DFPes-3)***$_\mathcal{L}$
$\circledast$ *satisfies* ***(DFPes-4)***$_\mathcal{L}$*, if and only if* $\kappa(\varphi) = \kappa(\psi)$
$\circledast$ *satisfies the* $\models$ *direction of* ***(DFPes-4)***$_\mathcal{L}$
$\circledast$ *falsifies* ***(DFPes-6)***$_\mathcal{L}$

For proofs and explanations of the relations stated in Th. 4.43 and 4.44, we refer to the elaborations above.

4.3.2 Further Connections between AGM Revisions, Iterated Revisions and Forgetting

In the following, we briefly examine some of the relations between the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ and the widely accepted revision postulates in order to check which revision postulates generally hold for forgetting operators as well. For the latter we consider the AGM revision postulates for epistemic states **(AGMes∗1)**-**(AGMes∗6)** (see Section 2.3 or Appendix A.1) on the one hand, and the postulates for iterated revision **(DP1)**-**(DP4)** (see Section 2.3 or Appendix A.1) as originally formulated by Darwiche and Pearl in [DP97] on the other. Since we already showed above that revision operators in the sense of the above-stated postulates do not satisfy the forgetting postulates in general, we further examine which of the revision postulates are satisfied by forgetting operators. Even though the examination of revision postulates for forgetting operators might not seem reasonable, due to the contrary success postulates, we want to investigate which of the forgetting operators relate to the revision postulates in order to gain further understanding on the relations between these two concepts. We do so by investigating the relations between the forgetting and the AGM revision postulates first, and afterwards we focus on the postulates for iterated belief revision. In the following, we show that forgetting operators do not satisfy most of the revision postulates, but that there also exist several commonalities, especially to the postulates for iterated revisions.

For the first AGM revision postulate **(AGMes∗1)**, which states the success of a belief revision, we already discussed above that it contradicts the success of a forgetting operator given by **(DFPes-6)**$_\mathcal{L}$ (Lem. 4.45).

Lemma 4.45. *Let Ψ be an epistemic state and $\varphi \in \mathcal{L}_\Sigma$ a formula. Further, let $\circ_f^{\mathcal{L}}$ be a belief change operator satisfying **(DFPes-6)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ contradicts*

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \varphi, \qquad \textbf{(AGMes}{*}\textbf{1)}$$

if $\varphi \not\equiv \top$.

The revision's success postulate **(AGMes∗1)** states that a certain formula can be inferred by the posterior beliefs, while the success of forgetting **(DFPes-6)$_{\mathcal{L}}$** states that a certain formula can no longer be inferred by the posterior beliefs, if it is non-tautologous.

For the second revision postulate **(AGMes∗2)** we know that it cannot be satisfied by forgetting operators either. The property described by this postulate says that the posterior beliefs after revising the epistemic state with φ is equivalent to the unification of the prior beliefs and φ unless the prior beliefs contradict φ. The fact that forgetting operators are not capable of satisfying **(AGMes∗2)** can again be traced back to the success postulate **(DFPes-6)$_{\mathcal{L}}$**.

Proposition 4.46. *Let Ψ be an epistemic state and $\varphi \in \mathcal{L}_\Sigma$ a formula. Further, let $\circ_f^{\mathcal{L}}$ be a belief change operator satisfying **(DFPes-6)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ contradicts*

$$\textit{if } Bel(\Psi) \cup \{\varphi\} \not\equiv \bot, \textit{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi) \cup \{\varphi\}, \qquad \textbf{(AGMes}{*}\textbf{2)}$$

if $\varphi \not\equiv \top$.

If we forget a non-tautologous formula φ in an epistemic state Ψ, then we know that φ can no longer be inferred by the posterior beliefs. Under the assumption that the prior beliefs do not contradict φ, we know that the unification of the prior beliefs with φ can especially infer φ. From this we can conclude that

$$\text{if } Bel(\Psi) \cup \{\varphi\} \not\equiv \bot, \text{ then } \underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi)}_{\not\models \varphi} \equiv \underbrace{Bel(\Psi) \cup \{\varphi\}}_{\models \varphi} \qquad \lightning$$

cannot be satisfied. Note that even if we assume $\circ_f^{\mathcal{L}}$ to satisfy the remaining forgetting postulates **(DFPes-1)$_{\mathcal{L}}$**-**(DFPes-5)$_{\mathcal{L}}$**, we cannot argue whether $\circ_f^{\mathcal{L}}$ satisfies **(AGMes∗2)** in case that $\varphi \equiv \top$, since the forgetting postulates do not prevent changes in the prior beliefs, when forgetting a tautology.

Next, we examine **(AGMes∗3)**, which says that a revision never results in contradictory beliefs, if the formula that the epistemic state is revised with is not contradictory. We will show that this postulate is satisfied by belief change operators satisfying **(DFPes-1)$_{\mathcal{L}}$** in case that the prior beliefs a non-contradictory as well, since we know due to **(DFPes-1)$_{\mathcal{L}}$** that the beliefs cannot be expanded due to forgetting (Prop. 4.47).

Proposition 4.47. *Let Ψ be an epistemic state, $\varphi \in \mathcal{L}_\Sigma$ a formula, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ satisfies*

$$\textit{if } \varphi \not\equiv \bot, \textit{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\equiv \bot, \qquad \textbf{(AGMes}{*}\textbf{3)}$$

if $Bel(\Psi) \not\equiv \bot$.

Proof of Prop. 4.47. In the following, we prove that $\circ_f^{\mathcal{L}}$ satisfies **(AGMes$*$3)**, if we assume the prior beliefs to be non-contradictory. Note that we show that under this assumption the consequence of **(AGMes$*$3)** holds in general, and thus especially for $\varphi \not\equiv \bot$.

$$\begin{aligned}
&\bot \not\equiv Bel(\Psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-1)}_{\mathcal{L}}\\
\Leftrightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \subseteq Bel(\Psi) \not\equiv \bot\\
\Leftrightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \subseteq Bel(\Psi) \not\equiv Cn(\bot) && \text{(Lem. 2.17)}\\
\Leftrightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \subseteq Bel(\Psi) \not\equiv \mathcal{L}_\Sigma && \text{(Lem. 2.20)}\\
\Leftrightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \subseteq Bel(\Psi) \subset \mathcal{L}_\Sigma\\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \subset \mathcal{L}_\Sigma\\
\Leftrightarrow\ & \mathcal{L}_\Sigma \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \text{ and } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \mathcal{L}_\Sigma\\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \mathcal{L}_\Sigma\\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models Cn(\bot)\\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \bot\\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\equiv \bot
\end{aligned}$$

□

The additional assumption $Bel(\Psi) \not\equiv \bot$ guarantees that the posterior beliefs are not contradictory, since the forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ do not prevent us from resulting in posterior beliefs that are equivalent to the prior, when forgetting a tautology. In this case it would be possible to result in contradictory beliefs.

In contrast to the other AGM revision postulates, **(AGMes$*$4)** is the only postulate that holds for forgetting operators satisfying **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ without further assumption, since the property stated by **(AGMes$*$4)** directly concludes from **(DFPes-2)**$_{\mathcal{L}}$ (Lem. 4.48).

Lemma 4.48. *Let Ψ and Φ be epistemic states, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying* ***(DFPes-2)***$_{\mathcal{L}}$*, then $\circ_f^{\mathcal{L}}$ satisfies*

$$\textit{if } \Psi = \Phi \textit{ and } \varphi \equiv \psi, \textit{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Phi \circ_f^{\mathcal{L}} \psi). \qquad \textbf{(AGMes}*\textbf{4)}$$

(DFPes-2)$_{\mathcal{L}}$ already states that applying $\circ_f^{\mathcal{L}}$ with a formula φ to two epistemic states with equivalent beliefs will result in equivalent posterior beliefs:

$$\text{if } Bel(\Psi) \equiv Bel(\Phi), \text{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Phi \circ_f^{\mathcal{L}} \varphi) \qquad \textbf{(DFPes-2)}_{\mathcal{L}}$$

The assumption $\Psi = \Phi$ stated in **(AGMes$*$4)** implies that the beliefs of both epistemic states must be equivalent, and since $\varphi \equiv \psi$ is further assumed, we know that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Phi \circ_f^{\mathcal{L}} \psi)$ holds especially.

Next, we show that **(AGMes$*$5)** is satisfied by forgetting operators in case that the posterior beliefs after forgetting φ, i.e. $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$, and φ do not contradict each other (Prop. 4.49).

Proposition 4.49. *Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$**,**(DFPes-3)$_{\mathcal{L}}$** and **(DFPes-6)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ satisfies*

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\} \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi), \qquad \textbf{(AGMes}{*}\textbf{5)}$$

if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi$ and $\psi \not\equiv \top$.

Proof of Prop. 4.49. Since $\varphi \wedge \psi \models \varphi$, we know due to **(DFPes-3)$_{\mathcal{L}}$** that

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \varphi \wedge \psi)$$

holds. Given this equivalence we can further conclude

$$\underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \varphi \wedge \psi)}_{\textbf{(DFPes-1)}_{\mathcal{L}}} \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \tag{4.9}$$

by means of **(DFPes-1)$_{\mathcal{L}}$**. The just shown relation stated in Eq. 4.9 can be shown for ψ analogously, since $\varphi \wedge \psi \models \psi$:

$$Bel(\Psi \circ_f^{\mathcal{L}} \psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi). \tag{4.10}$$

After forgetting ψ in Ψ, we are no longer able to infer ψ according **(DFPes-6)$_{\mathcal{L}}$**, and furthermore we know from Eq. 4.10, that the beliefs after forgetting $\varphi \wedge \psi$ can be inferred from those after forgetting ψ. Thus, we can conclude that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$ cannot infer ψ either:

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \not\models \psi.$$

Given the assumption $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi$, we further know that

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \not\models \neg\psi$$

also holds, because of $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$. Otherwise, this would contradict Eq. 4.10 shown above. Thus, we know that after forgetting $\varphi \wedge \psi$ neither ψ nor $\neg\psi$ can be inferred by the posterior beliefs. From this we can conclude that the unification of $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ and ψ can still infer $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ and is also able to infer $Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$:

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\} \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi).$$

Therefore, we proved that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\} \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$ holds under the assumption $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi$. □

The property described in Prop. 4.49 can be understood very intuitively, since we already know that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$ forgets more beliefs from Ψ than $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ (Eq. 4.9) and thus, we know that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$. When we now expand the beliefs $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ with ψ, then we are still able to infer $Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)$, since only new information were added to the beliefs.

Finally, we examine the last AGM revision postulate **(AGMes∗6)** and show that forgetting operators are not capable of satisfying this postulate either (Lem. 4.50).

Lemma 4.50. *Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-6)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ contradicts*

$$\begin{aligned}&\text{if } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\} \not\equiv \bot,\\ &\text{then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\},\end{aligned} \qquad \textbf{(AGMes}{*}\textbf{6)}$$

if $\varphi \not\equiv \top \not\equiv \psi$.

This can again be traced back to the contradicting success postulates **(AGMes$*$1)** and **(DFPes-6)$_{\mathcal{L}}$**. As already stated in Eq. 4.10, the posterior beliefs after forgetting $\varphi \wedge \psi$ can no longer infer ψ. On the other hand, we know that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\}$ is a contradiction if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \neg\psi$, or infers ψ if the beliefs do not contradict ψ, i.e. $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi$. From this we can conclude that

$$\underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi \wedge \psi)}_{\not\models \psi} \not\models \underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cup \{\psi\}}_{\equiv \bot \text{ or } \models \psi},$$

holds, which in fact contradicts **(AGMes$*$6)**.

We summarize the elaborated relations between the forgetting postulates **(DFPes-1)$_{\mathcal{L}}$**-**(DFPes-6)$_{\mathcal{L}}$** and the AGM revision postulates **(AGMes$*$1)**-**(AGMes$*$6)** in Th. 4.51.

Theorem 4.51. *Let $\circ_f^{\mathcal{L}}$ be a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$**-**(DFPes-3)$_{\mathcal{L}}$** and **(DFPes-6)$_{\mathcal{L}}$**, Ψ be an epistemic state and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas associated with **(AGMes$*$1)**-**(AGMes$*$6)**, then the following holds:*

$\circ_f^{\mathcal{L}}$ *contradicts **(AGMes$*$1)**, if $\varphi \not\equiv \top$*
$\circ_f^{\mathcal{L}}$ *contradicts **(AGMes$*$2)**, if $\varphi \not\equiv \top$*
$\circ_f^{\mathcal{L}}$ *satisfies **(AGMes$*$3)**, if $Bel(\Psi) \not\equiv \bot$*
$\circ_f^{\mathcal{L}}$ *satisfies **(AGMes$*$4)***
$\circ_f^{\mathcal{L}}$ *satisfies **(AGMes$*$5)**, if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi$ and $\psi \not\equiv \top$*
$\circ_f^{\mathcal{L}}$ *contradicts **(AGMes$*$6)**, if $\varphi \not\equiv \top \not\equiv \psi$*

For explanations and proofs of the stated relations, we refer to the elaborations above.

After we have examined the relations between the forgetting postulates and the AGM revision postulates for epistemic states, we further like to examine the relation to the postulates for iterated belief revision **(DP1)**-**(DP4)** (see Section 2.3 or Appendix A.1). For the first postulate **(DP1)**, we know that it is satisfied by any belief change operator satisfying the forgetting postulates **(DFPes-1)$_{\mathcal{L}}$**-**(DFPes-6)$_{\mathcal{L}}$**, since **(DP1)** is equivalent to **(DFPes-3)$_{\mathcal{L}}$** as already mentioned in Section 4.2. Also **(DP4)** holds for such operators $\circ_f^{\mathcal{L}}$ generally. This postulate originally states that a formula $\neg\psi$ cannot be inferred after revising an epistemic state Ψ with ψ and φ subsequently, if the revision with φ alone would already prevent concluding $\neg\psi$. When we transfer this property to the concept of forgetting, then it states that if forgetting φ is already sufficient to result in posterior beliefs not capable of inferring $\neg\psi$, then $\neg\psi$ cannot be inferred especially if we forgot ψ previous to φ. We formalize this property in Prop. 4.52.

Proposition 4.52. *Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$** and **(DFPes-5)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ satisfies*

$$\text{if } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi, \text{ then } Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \not\models \neg\psi. \qquad \textbf{(DP4)}$$

Proof of Prop. 4.52.

$$\begin{aligned} Bel(\Psi \circ_f^{\mathcal{L}} \varphi) &\models Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) && \textbf{(DFPes-1)}_{\mathcal{L}} \\ &\equiv Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) && \text{(Prop. 4.8)} \end{aligned}$$

Thus, we know that if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ does not infer $\neg\psi$, then $Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi)$ does not either. □

For the remaining two postulates **(DP2)** and **(DP3)**, we show that they do not hold for belief change operators satisfying **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**. For **(DP2)** we know that its consequence cannot hold in general, since this would require that the beliefs after forgetting two formulas φ and ψ consecutively are equivalent to the beliefs after forgetting φ only. There might exist cases in which this relation of the posterior beliefs holds. However, if we assume that ψ is non-tautologous and ψ could be inferred by the prior beliefs before forgetting it, we can show that **(DP2)** contradicts the assumed forgetting postulates. We formalize and proof this in Prop. 4.53 below.

Proposition 4.53. *Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$**, **(DFPes-5)$_{\mathcal{L}}$** and **(DFPes-6)$_{\mathcal{L}}$**, then $\circ_f^{\mathcal{L}}$ contradicts*

$$\text{if } \varphi \models \neg\psi, \text{ then } Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi), \qquad \textbf{\textit{(DP2)}}$$

if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \psi$ and $\psi \not\equiv \top$.

Proof of Prop. 4.53. In the following, we show that the consequence of **(DP2)** generally contradicts the properties stated in **(DFPes-1)$_{\mathcal{L}}$**, **(DFPes-5)$_{\mathcal{L}}$** and **(DFPes-6)$_{\mathcal{L}}$**, if we assume that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \psi$ holds, where ψ is non-tautologous.

$$\begin{aligned} & Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \\ \Leftrightarrow\ & Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \text{(Prop. 4.8)} \\ \Rightarrow\ & Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \text{(Def. 2.13)} \end{aligned}$$

Due to **(DFPes-6)$_{\mathcal{L}}$** and the assumption $\psi \not\equiv \top$, we can conclude that $Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \not\models \psi$ holds. Furthermore, we know by assumption that ψ can be inferred by $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$. Thus, we obtain

$$\underbrace{Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi)}_{\not\models \psi} \models \underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi)}_{\models \psi},$$

which shows that $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ cannot be inferred by $Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi)$, and therefore **(DP2)** contradicts the forgetting postulates **(DFPes-1)$_{\mathcal{L}}$**, **(DFPes-5)$_{\mathcal{L}}$** and **(DFPes-6)$_{\mathcal{L}}$**, in case that ψ is a non-tautologous formula that can be inferred before forgetting it. □

Lastly, **(DP3)** cannot be satisfied by operators $\circ_f^{\mathcal{L}}$ satisfying **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ either, since it contradicts both the success postulate **(DFPes-6)**$_{\mathcal{L}}$ and **(DFPes-1)**$_{\mathcal{L}}$. When transferring **(DP3)** to the concept of forgetting, its assumption reads that after forgetting formulas ψ and φ consecutively, ψ can still be inferred by the posterior beliefs. But due to **(DFPes-6)**$_{\mathcal{L}}$ this only holds, if ψ is a tautology. Thus, we know that **(DP3)** contradicts the assumed forgetting postulates, if ψ is non-tautologous (Prop. 4.54).

Proposition 4.54. *Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-5)** and **(DFPes-6)**, then $\circ_f^{\mathcal{L}}$ contradicts*

$$\textit{if } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \psi, \textit{ then } Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \models \psi, \qquad \textbf{(DP3)}$$

if $\psi \not\equiv \top$.

Proof of Prop. 4.54. In the following, we show that the consequence of **(DP3)** contradicts the properties stated by the forgetting postulates **(DFPes-5)**$_{\mathcal{L}}$ and **(DFPes-6)**$_{\mathcal{L}}$. By means of **(DFPes-5)**$_{\mathcal{L}}$, we know that $\circ_f^{\mathcal{L}}$ is commutative with respect to the beliefs of an epistemic state:

$$Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) \models \psi \Leftrightarrow Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \models \psi \qquad \text{(Prop. 4.8)}$$

At this point, we know due to **(DFPes-6)**$_{\mathcal{L}}$ and the assumption $\psi \not\equiv \top$ that

$$Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) \not\models \psi$$

must hold. Thus, **(DP2)** contradicts **(DFPes-5)** and **(DFPes-6)**, if ψ is assumed to be non-tautologous. □

Finally, we summarize the elaborated relations between the forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ and the postulates for iterated revision **(DP1)**-**(DP4)** in Th. 4.55.

Theorem 4.55. *Let Ψ be an epistemic state and $\varphi, \psi \in \mathcal{L}_\Sigma$ formulas referring to those in **(DP1)**-**(DP4)**. Further, let $\circ_f^{\mathcal{L}}$ be a belief change operator satisfying **(DFPes-1)**$_{\mathcal{L}}$, **(DFPes-3)**$_{\mathcal{L}}$, **(DFPes-5)**$_{\mathcal{L}}$, and **(DFPes-6)**$_{\mathcal{L}}$, then the following holds:*

$\circ_f^{\mathcal{L}}$ *satisfies **(DP1)***
$\circ_f^{\mathcal{L}}$ *contradicts **(DP2)**, if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \models \psi$ and $\psi \not\equiv \top$*
$\circ_f^{\mathcal{L}}$ *contradicts **(DP3)**, if $\psi \not\equiv \top$*
$\circ_f^{\mathcal{L}}$ *satisfies **(DP4)***

For proofs and explanations of these relations, we refer to the elaborated propositions Prop. 4.52 to 4.54 above.

Summary. In summary, we investigated the connections between propositional c-revisions and revisions satisfying **(AGMes$*$1)**-**(AGMes$*$6)**, respectively, and the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$. For this, we first stated that the implicit forgetting of such revisions can be explicated by means of contractions satisfying **(AGMes-1)**-**(AGMes-7)**, and thus in particular by means of minimal change c-contractions. Given this assumption, we could refer to the relations between **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ and contractions as stated in Th. 4.27 for the implicit forgetting of such revisions, and make use of them in order to investigate whether they also hold for such revisions, or how they change due to the consecutively performed expansion.

We were able to show that revisions satisfying **(AGMes$*$1)**-**(AGMes$*$6)**, and thus especially propositional c-revisions, contradict **(DFPes-1)**$_\mathcal{L}$ and **(DFPes-6)**$_\mathcal{L}$, since these postulates contradict the general notion of revision, which states that we want to be able to infer a new formula φ after revising with φ. However, in case of **(DFPes-6)**$_\mathcal{L}$ it is more appropriate to formulate this postulate with respect to $\neg\varphi$ instead of φ, since the implicit forgetting of the c-revision is applied to $\neg\varphi$. By doing so, we can conclude the fulfilment of **(DFPes-6)**$_\mathcal{L}$ for c-revisions. For **(DFPes-2)**$_\mathcal{L}$ we were able to show that c-revision do satisfy this postulate under the same additional restrictions as minimal change c-contractions and that the expansion performed by the revision has no influence on this property. Moreover, we showed that it is also possible to choose the revision parameters according to an appropriate strategy such that the refinement relation between two OCFs is retained after the revision. Other than minimal change c-contractions, both propositional c-revisions and revisions satisfying **(AGMes$*$1)**-**(AGMes$*$6)** satisfy **(DFPes-3)**$_\mathcal{L}$. This can mainly be traced back to the inverted relation of φ and ψ that is induced by the revision. For **(DFPes-4)**$_\mathcal{L}$ we showed that c-revisions only satisfy this postulate under further restrictions on the formulas we revise with. Furthermore, we showed that the $\models$ direction of **(DFPes-4)**$_\mathcal{L}$ holds in general. Finally, we showed that **(DFPes-5)**$_\mathcal{L}$ is neither satisfied by minimal change c-contractions nor by c-revisions, and that the additional restrictions necessary to satisfy **(DFPes-5)**$_\mathcal{L}$ cannot be transferred to c-revisions.

In addition to the examinations on the forgetting postulates for c-revisions, we investigated the relations between the forgetting postulates and both the AGM revision postulates for epistemic states **(AGMes$*$1)**-**(AGMes$*$6)** and the postulates for iterated belief revision **(DP1)**-**(DP4)**. There, we showed that the postulates **(AGMes$*$1)**, **(AGMes$*$2)**, **(AGMes$*$6)**, **(DP2)** and**(DP3)** are contradictory to the properties stated by the forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$, if we assume the occurring formulas to be non-tautologous. Moreover, we were able to show that **(AGMes$*$3)** and **(AGMes$*$5)** are satisfied under similar assumptions. For the remaining postulates **(AGMes$*$4)**, **(DP1)** and **(DP4)** we were able to show that they are satisfied without further assumptions. We summarized the elaborated relations in Th. 4.43, Th. 4.44 and Th. 4.55.

4.4 Properties of Forgetting

In the previous sections, we elaborated two generalized forms of Delgrande's forgetting postulates **(DFP-1)**-**(DFP-7)** (Th. 3.4, Appendix A.1). The first generalization **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ (Section 4.1, Appendix A.1) is a first attempt of applying Delgrande's notions of forgetting to arbitrary operators that result in epistemic states with a reduced signature. This was rather straightforward, since Delgrande's general forgetting approach was defined as a reduction of the signature as well. We were able to show that the OCF marginalization satisfies all of these postulates, and at the same time results in beliefs equivalent to the result of Delgrande's forgetting. After this, we generalized **(DFP-1)**-**(DFP-7)** such that they are not only applicable to epistemic states, but also to formulas instead of subsignatures. For this, we elaborated the fundamental ideas of these postulates, and chose appropriate representations in order to express these ideas with respect to formulas. However, while examining the second generalization **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ (Section 4.2, Appendix A.1) for c-contractions and c-revisions in Section 4.2 and Section 4.3, we noticed some rather controversial properties that are either directly stated or implied by them. In the following, we want to discuss these properties, arguing that the notions of forgetting as postulated by Delgrande are suitable for forgetting subsignatures, but not for formulas. Moreover, we prove that there cannot exist any useful belief change operator satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$. After this, we address the role of **(DFPes-3)**$_\mathcal{L}$ in this context, and suggest an adjusted variant that seems better suited for forgetting formulas.

The first controversial property implied by **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ is that forgetting a conjunction $\varphi \wedge \psi$ always results in beliefs that are not only incapable of inferring $\varphi \wedge \psi$, but also of inferring both φ and ψ (Prop. 4.56). This property can mainly be traced back to **(DFPes-3)**$_\mathcal{L}$.

Proposition 4.56. *Let Ψ be an epistemic state and $\circ_f^\mathcal{L}$ a belief change operator satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$, then*

$$Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) \not\models \varphi, \psi$$

holds for all formulas $\varphi, \psi \in \mathcal{L}$ with $\varphi \not\equiv \top \not\equiv \psi$.

Proof of Prop. 4.56.

$$\begin{aligned} Bel(\Psi \circ_f^\mathcal{L} \varphi) &\models Bel((\Psi \circ_f^\mathcal{L} \varphi) \circ_f^\mathcal{L} \varphi \wedge \psi) && \textbf{(DFPes-1)}_\mathcal{L} \\ &\equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) && \textbf{(DFPes-3)}_\mathcal{L} \end{aligned}$$

From **(DFPes-6)**$_\mathcal{L}$, we know that $Bel(\Psi \circ_f^\mathcal{L} \varphi) \not\models \varphi$, and since $Bel(\Psi \circ_f^\mathcal{L} \varphi) \models Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi)$, we know $Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) \not\models \varphi$ in conclusion. The same holds analogously, if we consider ψ instead of φ. Thus, we showed that the beliefs after forgetting $\varphi \wedge \psi$ cannot infer φ and ψ either. □

One can argue that the property stated in Prop. 4.56 yields unmotivated changes to the prior beliefs, since it is not always necessary to reject the beliefs about both φ and ψ in order to forget their conjunction. Au contraire, it is mostly sufficient to reject the beliefs about one of the formulas only, which could be the less plausible formula for example. This again shows that a forgetting operator according to **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ does not conform to the minimal change paradigm. However, this can also be viewed as a positive aspect, since minimizing propositional changes can induce undesired conditional changes in epistemic states [DP97]. Nevertheless, even if the property stated in Prop. 4.56 might be appropriate for some cognitive considerations, it is debatable if this property should be implied by postulates stating the general properties of forgetting formulas.

The next controversial property that is implied by **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$, is the fact that it does not matter, if we forget the conjunction or the disjunction of two formulas. Both posterior beliefs are equivalent (Prop. 4.57). Just as for Prop. 4.56, the implied property stated in Prop. 4.57 heavily depends on **(DFPes-3)**$_\mathcal{L}$.

Proposition 4.57. *Let Ψ be an epistemic state and $\circ_f^\mathcal{L}$ a belief change operator satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$, then*

$$Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) \equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \vee \psi)$$

holds for all formulas $\varphi, \psi \in \mathcal{L}$.

Proof of Prop. 4.57.

$$\begin{aligned} Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) &\equiv Bel((\Psi \circ_f^\mathcal{L} \varphi \vee \psi) \circ_f^\mathcal{L} \varphi \wedge \psi) && \textbf{(DFPes-3)}_\mathcal{L} \\ &\equiv Bel(\Psi \circ_f^\mathcal{L} (\varphi \wedge \psi) \vee (\varphi \vee \psi)) && \textbf{(DFPes-5)}_\mathcal{L} \\ &\equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \vee \psi) \end{aligned}$$

□

Forgetting $\varphi \vee \psi$ must result in posterior beliefs that are neither capable of inferring φ nor of inferring ψ. Thus, in order to result in equivalent beliefs as forgetting $\varphi \wedge \psi$, it is necessary for $Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi)$ to infer neither φ nor ψ as well. This shows that the equivalence stated in Prop. 4.57 is strongly related to Prop. 4.56.

Finally, we want to discuss the last controversial property we noticed during research. This property states that forgetting a conjunction $\varphi \wedge \psi$ must result in beliefs equivalent to just forgetting φ or ψ. (Prop. 4.58)

Proposition 4.58. *Let Ψ be an epistemic state and $\circ_f^\mathcal{L}$ a belief change operator satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$, then*

$$Bel(\Psi \circ_f^\mathcal{L} \varphi) \equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) \equiv Bel(\Psi \circ_f^\mathcal{L} \psi)$$

holds for all formulas $\varphi, \psi \in \mathcal{L}$.

Proof of (Prop. 4.58).

$$\begin{aligned} Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) &\equiv Bel((\Psi \circ_f^\mathcal{L} \varphi) \circ_f^\mathcal{L} \varphi \wedge \psi) && \textbf{(DFPes-3)}_\mathcal{L} \\ &\equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \vee (\varphi \wedge \psi)) && \textbf{(DFPes-5)}_\mathcal{L} \\ &\equiv Bel(\Psi \circ_f^\mathcal{L} \varphi) \end{aligned}$$

The above-stated equivalence holds for ψ analogously. Thus, we can conclude $Bel(\Psi \circ_f^\mathcal{L} \varphi) \equiv Bel(\Psi \circ_f^\mathcal{L} \varphi \wedge \psi) \equiv Bel(\Psi \circ_f^\mathcal{L} \psi)$. □

Furthermore, Prop. 4.58 implies that forgetting always results in equivalent beliefs, independent of which formulas we forget (Cor. 4.59).

Corollary 4.59. *Let Ψ be an epistemic state and $\circ_f^{\mathcal{L}}$ a belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**, then*

$$Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \psi)$$

holds for all formulas $\varphi, \psi \in \mathcal{L}$.

With this obviously being an undesired behaviour, since it means that forgetting works independently of what we would like to forget, it raises the question, whether such a belief change operator capable of satisfying **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$** can exist at all. From **(DFPes-1)$_{\mathcal{L}}$**, we know that the prior beliefs cannot be extended due to forgetting. Moreover, we know due to **(DFPes-6)$_{\mathcal{L}}$**, that after forgetting a certain non-tautologous formula, we are no longer able to infer it. Nonetheless, since forgetting is independent of the formula we forget (Cor. 4.59), we know that once we apply $\circ_f^{\mathcal{L}}$ to an epistemic state, we are no longer allowed to infer anything, but tautologies. Thus, a belief change operator always resulting in posterior beliefs equivalent to $\top$ can be the only belief change operator satisfying **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$** (Th. 4.60).

Theorem 4.60 (Triviality Result)**.** *Let Ψ be an epistemic state. A belief change operator $\circ_f^{\mathcal{L}}$ satisfies **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**, if and only if $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv \top$ holds for each $\varphi \in \mathcal{L}$.*

Proof of Th. 4.60. We prove Th. 4.60 in two steps. Firstly, we show that if a belief change operator satisfies **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**, then it must always result in posterior beliefs $Bel(\Psi \circ_f^{\mathcal{L}} \varphi)$ equivalent to $\top$. Secondly, we show that each belief change operator $\circ_f^{\mathcal{L}}$ with $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv \top$ for each $\varphi \in \mathcal{L}$ satisfies **(DFPes-1)$_{\mathcal{L}}$-(DFPes-6)$_{\mathcal{L}}$**. We refer to these two steps as $(\Rightarrow)$ and $(\Leftarrow)$. Note that we assume all formulas $\varphi, \psi \in \mathcal{L}$ to be non-tautologous.

Case $(\Rightarrow)$: From Cor. 4.59, we know that applying $\circ_f^{\mathcal{L}}$ to an epistemic state Ψ must result in equivalent beliefs for all formulas $\varphi, \psi \in \mathcal{L}$. From **(DFPes-6)$_{\mathcal{L}}$** we know that after forgetting a formula φ, we are no longer able to infer φ. Since the posterior beliefs are equivalent for all formulas, we can conclude that after applying $\circ_f^{\mathcal{L}}$ to Ψ, we are not able to infer any formula, but tautologies.

$$\begin{aligned}
& Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \psi), \text{ for all } \varphi, \psi \in \mathcal{L} && \text{(Cor. 4.59)} \\
\Rightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \varphi, \psi, \text{ for all } \varphi, \psi \in \mathcal{L} && \textbf{(DFPes-1)}_{\mathcal{L}} \\
\Leftrightarrow\ & Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv \top, \text{ for all } \varphi \in \mathcal{L}
\end{aligned}$$

Case $(\Leftarrow)$: Let Ψ and Φ be epistemic states and $\varphi, \psi \in \mathcal{L}$ be non-tautologous formulas, and $\circ_f^{\mathcal{L}}$ a belief change operator with $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv \top$ for all epistemic states Ψ and formulas φ. Further, we refer to the assumption $Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv \top$ as $(\top)$.

(DFPes-1)$_{\mathcal{L}}$:

$$\begin{aligned} & Bel(\Psi) \models Bel(\Psi \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-1)}_{\mathcal{L}} \\ \Leftrightarrow\ & Bel(\Psi) \models \top && (\top) \end{aligned}$$

Since $Bel(\Psi) \models \top$ holds in general, $\circ_f^{\mathcal{L}}$ satisfies **(DFPes-1)$_{\mathcal{L}}$**.

(DFPes-2)$_{\mathcal{L}}$:

$$\begin{aligned} & \text{if } Bel(\Psi) \equiv Bel(\Phi), \text{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel(\Phi \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-2)}_{\mathcal{L}} \\ \Leftrightarrow\ & \text{if } Bel(\Psi) \equiv Bel(\Phi), \text{ then } \top \equiv \top && (\top) \end{aligned}$$

Since $\top \equiv \top$ holds trivially, it especially holds if $Bel(\Psi) \equiv Bel(\Phi)$. Thus, $\circ_f^{\mathcal{L}}$ satisfies **(DFPes-2)$_{\mathcal{L}}$**.

(DFPes-3)$_{\mathcal{L}}$:

$$\begin{aligned} & \text{if } \varphi \models \psi, \text{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \equiv Bel((\Psi \circ_f^{\mathcal{L}} \psi) \circ_f^{\mathcal{L}} \varphi) && \textbf{(DFPes-3)}_{\mathcal{L}} \\ \Leftrightarrow\ & \text{if } \varphi \models \psi, \text{ then } \top \equiv \top && (\top) \end{aligned}$$

Since $\top \equiv \top$ holds trivially, it especially holds if $\varphi \models \psi$. Thus, $\circ_f^{\mathcal{L}}$ satisfies **(DFPes-3)$_{\mathcal{L}}$**.

(DFPes-4)$_{\mathcal{L}}$:

$$\begin{aligned} & Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) \equiv Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \cap Bel(\Psi \circ_f^{\mathcal{L}} \psi) && \textbf{(DFPes-4)}_{\mathcal{L}} \\ \Leftrightarrow\ & \top \equiv \underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \varphi)}_{\equiv \top} \cap \underbrace{Bel(\Psi \circ_f^{\mathcal{L}} \psi)}_{\equiv \top} && (\top) \\ \Leftrightarrow\ & \top \equiv \{\varphi \in \mathcal{L} \mid \top \models \varphi\} \cap \{\varphi \in \mathcal{L} \mid \top \models \varphi\} \\ \Leftrightarrow\ & \top \equiv \{\varphi \in \mathcal{L} \mid \top \models \varphi\} \\ \Leftrightarrow\ & \top \equiv \top \end{aligned}$$

(DFPes-5)$_{\mathcal{L}}$:

$$\begin{aligned} & Bel(\Psi \circ_f^{\mathcal{L}} \varphi \vee \psi) \equiv Bel((\Psi \circ_f^{\mathcal{L}} \varphi) \circ_f^{\mathcal{L}} \psi) && \textbf{(DFPes-5)}_{\mathcal{L}} \\ \Leftrightarrow\ & \top \equiv \top && (\top) \end{aligned}$$

(DFPes-6)$_{\mathcal{L}}$:

$$\begin{aligned} & \text{if } \varphi \not\equiv \top, \text{ then } Bel(\Psi \circ_f^{\mathcal{L}} \varphi) \not\models \varphi && \textbf{(DFPes-6)}_{\mathcal{L}} \\ \Leftrightarrow\ & \text{if } \varphi \not\equiv \top, \text{ then } \top \not\models \varphi && (\top) \end{aligned}$$

We showed that both cases ($\Rightarrow$) and ($\Leftarrow$) hold, and therefore proved the triviality result stated in Th. 4.60. $\square$

Th. 4.60 clearly shows, that **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ are not appropriate for stating general properties of forgetting formulas. However, all of the above-mentioned controversial and undesired properties of forgetting can mainly be traced back to **(DFPes-3)**$_{\mathcal{L}}$. Thus, we further want to discuss the third forgetting postulate **(DFPes-3)**$_{\mathcal{L}}$ for forgetting formulas in epistemic states, which corresponds to Delgrande's fourth postulate **(DFP-4)**. For the latter, we know that forgetting a subsignature P will result in equivalent beliefs as forgetting two subsignatures P and P' subsequently, if we assume $P' \subseteq P$. In Section 4.2 we tried to reflect this property as accurate as possible when generalizing the postulates to epistemic states and formulas. However, our examinations on the postulates for c-contractions and c-revisions in Sections 4.2 and 4.3 revealed that expressing the subset relation $P' \subseteq P$ by means of $\varphi \models \psi$, where P' is more specific than P and φ is more specific than ψ, admittedly captures the idea of the original postulate **(DFP-4)**, but affects the prior beliefs other than expected. In Section 4.2, we showed that minimal change c-contractions do not satisfy **(DFPes-3)**$_{\mathcal{L}}$ (Prop. 4.22) in general. When we examined the same postulate for c-revisions in Section 4.3, we showed that c-revisions are actually capable of satisfying **(DFPes-3)**$_{\mathcal{L}}$ (Prop. 4.39). There, we further discussed that the reason for this lays in the inverted relation of the formulas φ and ψ. Due to the postulate we assumed $\varphi \models \psi$ for the c-revisions, but since the implicit forgetting of c-revisions is applied to $\neg\varphi$ and $\neg\psi$ instead, we actually performed the forgetting of the more specific information first, and the forgetting of the more general information afterwards, and compared the posterior beliefs to those after just forgetting the more general information. The following example Ex. 4.12 will intuitively illustrate why the inverted relation of φ and ψ seems to be more appropriate for forgetting formulas.

Example 4.12. *In this example, we want to illustrate why assuming $\psi \models \varphi$ instead of $\varphi \models \psi$ in **(DFPes-3)**$_{\mathcal{L}}$ is more appropriate for the forgetting of formulas. For this, we consider the following scenario.*

We are software developers working in our office in a software company. When we tried to print some important documents, we recognized that the printer is not working, because there is no paper left ($\neg p$) and the cartridges are empty ($\neg c$) as well. Thus, our beliefs about the printer at this moment can be described as $Bel(\Psi) \equiv \neg p \wedge \neg c$. Since it is almost time for lunch, we decide to go to the canteen first, and print the documents later. Approximately one hour later, we come back to our office. Since almost an whole hour has passed, we are no longer sure, if the printer is still not ready for use. Therefore, we forget our belief about the printer not being ready for use, i.e. we forget $\neg p \vee \neg c$, since the printer cannot be used, if there is no paper left ($\neg p$) or the cartridge is empty ($\neg c$). Thus, we do not know anything about the printer's status: $Bel(\Psi^{\circ}) \equiv \top$. Suddenly, we see some co-worker with a big stack of paper moving in the direction of the printer room. Maybe the co-worker is re-filling the paper of the printer. At this point, we would actually forget that the paper is gone ($\neg p$), which is a more specific information than $\neg p \vee \neg c$, because $\neg p \models \neg p \vee \neg c$. But since we are already unsure about the printer's status, forgetting $\neg p$ does not influence our current beliefs about the printer. Thus, when forgetting the less specific information $\neg p \vee \neg c$ first, forgetting the more specific information $\neg p$

has no influence on our beliefs. This means that our posterior beliefs are equivalent to those, when just forgetting $\neg p \vee \neg c$.

As intuitively illustrated in Ex. 4.12, it seems more appropriate to invert the relation of φ and ψ stated in **(DFPes-3)**$_{\mathcal{L}}$, since forgetting a more general information also affects our beliefs about more specific information. Therefore, we suggest to change **(DFPes-3)**$_{\mathcal{L}}$ to:

(DFPes-3)$^{\star}_{\mathcal{L}}$ If $\varphi \models \psi$, then $Bel(\Psi \circ^{\mathcal{L}}_{f} \psi) \equiv Bel((\Psi \circ^{\mathcal{L}}_{f} \varphi) \circ^{\mathcal{L}}_{f} \psi)$

This suggestion is strengthened by the fact that when replacing **(DFPes-3)**$_{\mathcal{L}}$ by **(DFPes-3)**$^{\star}_{\mathcal{L}}$, none of the above-stated unmotivated properties Prop. 4.56 to 4.58 and Cor. 4.59 is implied by the forgetting postulates anymore. If we assume a belief change operator $\circ^{\mathcal{L}}_{f}$ to satisfy **(DFPes-3)**$^{\star}_{\mathcal{L}}$ instead of **(DFPes-3)**$_{\mathcal{L}}$, we can no longer conclude that Prop. 4.56 holds in general, since it is not possible to show $Bel(\Psi \circ^{\mathcal{L}}_{f} \varphi) \models Bel(\Psi \circ^{\mathcal{L}}_{f} \varphi \wedge \psi)$ without **(DFPes-3)**$_{\mathcal{L}}$. On the other hand, it is not implicitly excluded by the postulates. The fact that it is not possible to conclude Prop. 4.56 without **(DFPes-3)**$_{\mathcal{L}}$ further implies that it is not possible to satisfy Prop. 4.57 either, since this would require that after forgetting a conjunction $\varphi \wedge \psi$ neither φ nor ψ can be inferred anymore. Finally, changing **(DFPes-3)**$_{\mathcal{L}}$ to **(DFPes-3)**$^{\star}_{\mathcal{L}}$ also hinders us from concluding that the property stated in Prop. 4.58 holds in general. As a result, we know that Cor. 4.59 and Th. 4.60 do not hold either.

In conclusion, we illustrate some of the controversial properties that are implied by the forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$, and argued that they can mainly be traced back to **(DFPes-3)**$_{\mathcal{L}}$. Moreover, we showed that a belief change operator satisfies **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$, if and only if it results in a posterior epistemic state with beliefs equivalent to $\top$. Due to this, we discussed that **(DFPes-3)**$_{\mathcal{L}}$ appropriately captures the idea stated by the corresponding original postulate **(DFP-4)**, but argued that other than for forgetting subsignatures, it is not suitable for forgetting formulas. Therefore, we suggested an adjusted third forgetting postulate **(DFPes-3)**$^{\star}_{\mathcal{L}}$, better suitable for forgetting formulas, and argued that none of the here stated controversial and undesired properties hold, if we replace **(DFPes-3)**$_{\mathcal{L}}$ with **(DFPes-3)**$^{\star}_{\mathcal{L}}$. Even if this seems to legitimize the replacement of **(DFPes-3)**$_{\mathcal{L}}$, we think further research on **(DFPes-3)**$^{\star}_{\mathcal{L}}$ is required in order to decide whether it is really appropriate as a forgetting postulate. In addition to this, the influence of the remaining postulates to more controversial and unmotivated implied properties should be investigated, as well.

5 Conclusion and Future Work

Summary and Conclusion. In this work, we presented and elaborated the definitions and kinds of forgetting as given in [Del17] by Delgrande, and in [BKIS$^+$19] by Kern-Isberner et al. These works are two of the most recent works towards the generalization of forgetting in knowledge representation. In Section 3.1, we elaborated Delgrande's general approach of forgetting, which is capable of expressing several of the hitherto logic-specific forgetting approaches, such as forgetting atoms in propositional logic [Boo54], or forgetting predicate symbols and constants in first-order logic [LR94, ZZ10]. In contrast to most of the logic-specific approaches, Delgrande's definition (Def. 3.1) performs forgetting on the *knowledge level* instead of regarding the syntactic appearance of the knowledge. With forgetting on the *knowledge level* Delgrande refers to the deductive closure of a given set of formulas, and its corresponding models. Thus, forgetting is always performed with respect to all the knowledge that can be inferred priorly, and again results in a deductively closed set of formulas that are believed afterwards. This way it is possible to apply this definition to any logic with some realisation of the concept of deductive reasoning, for example by means of a Tarskian consequence relation (Def. 2.12) or a consequence operator Cn (Def. 2.16). Thereby, Delgrande defines forgetting as a reduction of the language or signature, respectively, which means that the prior conclusions we want to forget are given by means of a subsignature instead of a set of formulas. Thus, Delgrande considers forgetting as forgetting objects and concepts of our worlds, such that it is not possible to argue about them afterwards. This is guaranteed, by the fact that this forgetting approach results in a reduced language, which does not contain formulas mentioning elements of the forgotten subsignature. However, this also shows that Delgrande's general forgetting approach is not capable of forgetting specific facts from our prior beliefs. Delgrande argues that the removal of facts from our beliefs should not be considered as forgetting, since it is conceptually different from their idea of forgetting. Instead, they argue that the removal of facts corresponds the the concept of contraction. At this point, Delgrande claims the term *forgetting* for their own definition. However, we disagree with this point of view, since we think that Delgrande's approach does not capture the whole variety of different cognitive kinds of forgetting, but rather describes one of them.

Moreover, we elaborated the most important properties of Delgrande's forgetting approach, and referred to them as postulates **(DFP-1)**-**(DFP-7)** in this work. These postulates are of particular interest, since they seem to state properties that are generally applicable to the concept of forgetting, even though they are explicitly formulated with respect to Delgrande's definition. Additionally, Delgrande emphasizes that these properties are *right*, in the sense that they correspond to the intuitive idea of forgetting. Therefore, we used **(DFP-1)**-**(DFP-7)** as a starting point for our elaborations towards more general forgetting postulates in Section 4.1 and Section 4.2.

In Section 3.2, we elaborated three of the several kinds of forgetting presented by Kern-Isberner et al. in [BKIS$^+$19]. This is different to Delgrande's attempt of unifying the existing logic-specific approaches, since the major goal of Kern-Isberner et al. is the elaboration and axiomatization of cognitively different kinds of forgetting. Instead of deductively closed sets of formulas, the there presented kinds of forgetting argue about epistemic states, and therefore are even more general, because this theoretically allows us to apply them to any kind of chosen knowledge representation, e.g. probability distributions, Markov chains or other statistical and machine learning models. The three kinds of forgetting we decided to elaborate in this work are the marginalization (Section 3.2.1), the contraction (Section 3.2.2) and the revision (Section 3.2.3). In our opinion, these are the most important kinds of forgetting presented by Kern-Isberner et al., because of the following reasons. The marginalization forms the only cognitive kind of forgetting stated in [BKIS$^+$19] that argues about forgetting concepts and objects of our worlds, i.e. signature elements, instead of beliefs about them, and therefore is conceptually similar to Delgrande's definition of forgetting. The contraction, as well as the revision, forms two of the most fundamental and important concepts of belief change in the domain of knowledge representation, and are subject of many researches, not least because they are part to the well-established AGM theory [AGM85, Mak88, GR95]. Moreover, the revision forms the only kind of forgetting that does not explicitly state the removal of any prior beliefs, since its success postulate states that after revising with a certain formula, it should be included in the posterior beliefs. The forgetting at this point is of implicit nature, since successfully incorporating a new information might require to give up some of the prior beliefs that contradict it. Concretely, we examined contractions and revisions by means of c-contractions and c-revisions. An interesting observation we made during our elaboration of the marginalization is its influence on conditional beliefs, when we lift an marginalized OCF back to its original signature. In this case, the resulting OCF is not able to infer non-trivial propositions mentioning the just forgotten signature elements, while it is possible to infer non-trivial conditional beliefs mentioning the just forgotten signature elements that could not be inferred by the prior beliefs (Obs. 3.34). Thus, marginalizing and lifting an OCF might yield new conditional beliefs.

In Section 4, we then generalized and extended the properties of Delgrande's forgetting approach **(DFP-1)**-**(DFP-7)** as a first attempt of postulating general properties for different kinds of forgetting that are beyond those stated in [BKIS$^+$19]. Thereby, we elaborated two sets of postulates. The first set of postulates **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ states properties of forgetting signature elements, while the second set **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ states properties of forgetting formulas. This differentiation is necessary, since these both kinds of forgetting are conceptually different. At this point, we agree with Delgrande when they argue that the removal of certain facts is different to their idea of forgetting. Furthermore, we examined the different kinds of forgetting with respect to the generalized forgetting postulates.

In Section 4.1, we first showed that the marginalization results in beliefs equivalent to the result of Delgrande's forgetting approach (Th. 4.1). Thus, we know that Delgrande's approach is also covered by the different kinds of forgetting presented by Kern-Isberner et al., which further supports our position that Delgrande's approach just represents one kind of forgetting, rather than stating a comprehensive general forgetting definition. Moreover, we showed that the marginalization satisfies all of the generalized forgetting postulates **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ (Th. 4.6), which is a strong evidence that these postulates are suitable for stating general properties of forgetting signature elements. Thereby, we also showed that the marginalization is of particular importance for the concept of forgetting signature elements. Due to **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, we were able to show that the changes induced to an epistemic state and its corresponding beliefs are bounded below by the changes of the marginalization. This means that each other operator satisfying **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$ at least induces the same changes as the marginalization (Prop. 4.7). We also illustrated this by means of two concrete examples (Ex. 4.1). Furthermore, we were able to show that all of the model theoretical considerations that hold for Delgrande's approach, also hold for the marginalization (Th. 4.3, Cor. 4.4).

In Section 4.2, we stated a second set of forgetting postulates **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ that generalizes and extends the properties stated in **(DFP-1)**-**(DFP-7)**, but in contrast to **(DFPes-1)**$_\Sigma$-**(DFPes-6)**$_\Sigma$, in a way that they argue about formulas instead of signature elements. Thus, we formulated two sets of forgetting postulates in total, distinguishing between the notions of forgetting concepts and objects of worlds, and forgetting beliefs and facts about the latter. At this point, we already noticed some commonalities to some of the already established belief change postulates in the domain of knowledge representation, namely the equivalence of **(DFPes-1)**$_\mathcal{L}$ and **(AGMes-1)**, **(DFPes-3)**$_\mathcal{L}$ and **(DP1)**, as well as **(DFPes-6)**$_\mathcal{L}$ and **(AGMes-3)**. Further, we showed that c-contractions are not capable of satisfying **(DFPes-1)**$_\mathcal{L}$-**(DFPes-6)**$_\mathcal{L}$ in general (Th. 4.9), with the exception of **(DFPes-6)**$_\mathcal{L}$ which exactly corresponds to the success postulate of contraction as stated in [BKIS+19] and **(AGMes-3)**. We noticed that this can be traced back to the way c-contraction are allowed to affect the prior beliefs. Since c-contractions in general are not contractions in the sense of AGM, they do not necessarily correspond to the minimal change paradigm. Concretely, any c-change operator that guarantees the contracted formula to be not inferable by the posterior beliefs is considered a c-contraction. This makes it difficult to examine general properties of c-contractions.

With this not being sufficient to satisfy **(DFPes-1)**$_\mathcal{L}$-**(DFPes-5)**$_\mathcal{L}$, we further examined the forgetting postulates for those c-contractions that only induce minimal changes to the prior beliefs, and therefore satisfy the AGM contraction postulates for epistemic states **(AGMes-1)**-**(AGMes-7)**. We showed that such c-contractions, which we referred to as minimal change c-contractions (Def. 3.37), do not generally satisfy the forgetting postulates either, with the exception of **(DFPes-1)**$_\mathcal{L}$ and **(DFPes-6)**$_\mathcal{L}$ (Th. 4.27). However, we elaborated further conditions under which the postulates are satisfied. These additional conditions make further assumptions about the relation of the formulas that should be forgotten, as well as their minimal models. Most interestingly, we showed that **(DFPes-2)**$_\mathcal{L}$ strongly relates to the concept of refinement, which is sufficient in order to satisfy **(DFPes-2)**$_\mathcal{L}$ (Prop. 4.16). As part of the examinations for minimal change c-contractions and **(DFPes-2)**$_\mathcal{L}$, we elaborated the minimal model subset relation $\subseteq_{\min,\kappa}$ (Def. 4.17) that states how minimal models relate to each other within an OCF κ. Visualizing $\subseteq_{\min,\kappa}$ by means of a Hasse diagrams (see Figure 3) allows us to gain a better understanding of the minimal model relations. Furthermore, this visualization can also be used to understand how differences in multiple OCFs affect the relations of the minimal models more easily. We think that this is also a promising approach for visualizing the effects of belief changes in future works. However, since these visualizations quickly become very complex with an increasing number of signature elements, they are most useful when arguing about OCFs with rather small signatures. This problem could be tackled in future works with different and more appropriate visualization approaches, such as unifying the cliques in the Hasse diagrams to a single node.

After this, we showed that the **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ imply all of the contraction postulates **(AGMes-1)**-**(AGMes-7)**, except for **(AGMes-2)** and the recovery postulate **(AGMes-4)** (Th. 4.51). Nonetheless, the fulfilment of **(AGMes-2)** and **(AGMes-4)** is not excluded by the forgetting postulates either. In our opinion, this illustrates that the concepts of forgetting and contraction are compatible, which again disagrees to Delgrande's statement that contractions should not be considered as forgetting. The fact that especially **(AGMes-2)** and **(AGMes-4)** are not implied by **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ shows that forgetting formulas according to these postulates does not necessarily correspond to the minimal change paradigm. This can be a desired behaviour when arguing about iterated and conditional belief changes as discussed by Darwiche and Pearl [DP97].

Finally, we showed in Section 4.3 that the implicit forgetting of c-revisions can be realised by any contraction satisfying **(AGMes-1)**-**(AGMes-7)**, and thus especially by minimal change c-contractions. Therefore, we elaborated how the relations of minimal change c-contractions and **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ behave for c-revision. We showed that most of the conditions sufficient for contractions to satisfy the forgetting postulates cannot be transferred to revisions, mainly because the revision with a formula φ contracts $\neg\varphi$, while the postulates still argue about φ. Most notably, the refinement relation assumed for **(DFPes-2)**$_{\mathcal{L}}$ and minimal change c-contraction is also sufficient for c-revisions to satisfy **(DFPes-2)**$_{\mathcal{L}}$ (Prop. 4.37). In contrast to minimal change c-contractions, c-revisions are satisfy **(DFPes-3)**$_{\mathcal{L}}$ in general (Prop. 4.39), which is again due to the fact that revisions argue about forgetting $\neg\varphi$, while the forgetting postulates argue about φ itself. We were also able to prove that some of the relations between c-revisions and **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ also hold for general AGM revisions (Th. 4.43), since they only depend on the way the revision affects the most plausible interpretations. However, the other relation could hold for AGM revisions as well. For this further elaborations about general epistemic states would be necessary. During the examinations of the forgetting postulates and c-revisions, we noticed that it might be more appropriate to consider revisions with φ as forgetting $\neg\varphi$, since many of the here elaborated relations are based on the fact that the implicit forgetting of revisions concerns $\neg\varphi$, while the forgetting postulates still argue about φ when applied straightforwardly to revisions. Therefore, we suggest to consider revisions with φ as forgetting $\neg\varphi$ in future works, whenever arguing about revisions in the context of forgetting. Afterwards, we examined the connection between the forgetting postulates and the AGM revision postulates **(AGMes$*$1)**-**(AGMes$*$6)**, as well as the postulates for iterated revision **(DP1)**-**(DP4)**. We showed that **(AGMes$*$1)**,**(AGMes$*$2)** and **(AGMes$*$6)** are contradicted by the forgetting postulates, if we assume the formulas they argue about to be non-tautologous. On the other hand, we showed that **(AGMes$*$3)** and **(AGMes$*$5)** hold under further assumptions, and that **(AGMes$*$4)** is the only postulate that is implied by **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ (Th. 4.51). For the postulates of iterated revision, we showed that both **(DP1)** and **(DP4)** are implied by the forgetting postulates, while **(DP2)** and **(DP3)** are contradicted when we again assume the formulas to be non-tautologous (Th. 4.55).

During our examinations on the generalized forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$, we noticed several controversial properties (Prop. 4.56 to 4.58 and Cor. 4.59) that are implied by these postulates, and discussed them in Section 4.4. All of them could be traced back to the third forgetting postulate **(DFPes-3)**$_{\mathcal{L}}$. Furthermore, we were able to show that each operator satisfying **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ must always result in posterior beliefs that only consist of tautologies (Th. 4.60). This mainly goes back to the implied property that the result of forgetting must be independent of the formula we like to forget. Due to this triviality result and our insight on **(DFPes-3)**$_{\mathcal{L}}$ from our examinations for c-revisions, we suggested an alternative to **(DFPes-3)**$_{\mathcal{L}}$, namely **(DFPes-3)**$^{\star}_{\mathcal{L}}$, which states that forgetting a more specific and a more general information consecutively must always result in beliefs equivalent to just forgetting the more general information. Replacing **(DFPes-3)**$_{\mathcal{L}}$ by **(DFPes-3)**$^{\star}_{\mathcal{L}}$ should prevent the forgetting from being independent of the formula we like to forget, and thus the triviality result as well. However, its validation is still pending and could be subject in future works.

In conclusion, we believe that the here presented postulates for forgetting signature elements **(DFPes-1)**$_{\Sigma}$-**(DFPes-6)**$_{\Sigma}$ are actually suitable for describing general properties for this kind of forgetting, since the marginalization satisfies all of them and at the same time defines a lower bound for the changes each other operator satisfying these postulates induces to the prior beliefs. This is consistent with the idea that the marginalization forms the signature forgetting that induces only minimal changes to the prior beliefs as indirectly given in [BKIS$^+$19], where it is stated that a marginalized OCF is capable of inferring all of the prior propositional and conditional beliefs that are defined over the reduced signature. Moreover, this corresponds to the here shown equivalence between the result of Delgrande's forgetting approach and the beliefs of a marginalized OCF. In contrast to that, we believe that the here presented first attempt of postulating general properties for forgetting formulas is not yet suitable, not least because of the triviality result we elaborated in our researches that states that an operator satisfying **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ must always result in tautologous beliefs. This might be due to the fact that we used the properties **(DFP-1)**-**(DFP-7)** stated for Delgrande's forgetting approach, which clearly can be considered as the forgetting of signature elements, as a basis for the elaboration of postulates for forgetting formulas. This supports our assumption that it is necessary to state different sets of postulates for these two kinds of forgetting, since they are conceptually different. Nevertheless, we believe that **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$ form a good basis for further research on general properties for forgetting formulas, especially because of the shown relations to the AGM contraction postulates **(AGMes-1)**-**(AGMes-7)**, which are all implied the forgetting postulates, except for those that enforce the minimal change paradigm.

With the alternative variant of the third forgetting postulate, namely **(DFPes-3)**$^{\star}_{\mathcal{L}}$, we paved the way for future works, since we believe that the difference between **(DFPes-3)**$_{\mathcal{L}}$ and **(DFPes-3)**$^{\star}_{\mathcal{L}}$ illustrates one of the essential differences between the forgetting of signature elements and the forgetting of formulas.

Future Work. Since the research towards a general framework for forgetting just emerged in the last few years, the possible directions and open questions that can be approached in future works are numerous. In the following, we state some of the open questions that emerged from our examinations in this work, and furthermore some general issues that could be covered in future works.

In our conclusion above, we already mentioned some open questions that could be examined in future works. These concern among others how forgetting, or belief changes in general, affect the relations of the minimal models within an epistemic state. The changes could be visualized by means of Hasse diagrams and the minimal subset relation $\subseteq_{\min,\cdot}$, such that they can be understood and examined more easily. However, since Hasse diagrams become rather complex with an increasing number of signature elements, the elaboration of a more appropriate visualization might be necessary. Moreover, the relations Delgrande states between their definition of forgetting and the logic-specific approaches in [Del17] should further be examined in the general framework presented by Kern-Isberner et al. [BKIS^{+}19], since the here shown equivalence of the marginalization and Delgrande's approach with respect to the resulting beliefs (Th. 4.1) reveals that these relations should hold for the marginalization as well. Thus, it seems promising that these relations can be embedded in the general framework presented in [BKIS^{+}19]. Concerning the suggested revised third forgetting postulate **(DFPes-3)**$^{\star}_{\mathcal{L}}$, further examinations on the impact of **(DFPes-3)**$^{\star}_{\mathcal{L}}$ could be covered in future works, including which new properties are implied by the postulates, and if the triviality result (Th. 4.60) still holds when replacing **(DFPes-3)**$_{\mathcal{L}}$ by **(DFPes-3)**$^{\star}_{\mathcal{L}}$. Moreover, it is still to be examined if the remaining postulates should be revised as well.

So far, we only covered three of the several kinds of forgetting presented in [BKIS^{+}19]. Thus, it might be interesting to further elaborate the remaining kinds of forgetting, as well as their relations to each other, especially with respect to the generalized forgetting postulates. Since we mostly considered forgetting in a propositional framework, further examinations on the influence of forgetting on conditional beliefs are necessary. In this context, it could be interesting to revise the relevance of refinements, and elaborate if the here stated relation of refinements and the posterior beliefs (Prop. 4.16 and 4.37) also hold for conditional beliefs and arbitrary epistemic states, i.e. if

$$\text{if } \Psi \sqsubseteq \Phi, \text{ then } \Psi^{\circ} \mathrel{|\!\!\approx} (\psi|\varphi) \Rightarrow \Phi^{\circ} \mathrel{|\!\!\approx} (\psi|\varphi)$$

holds for epistemic states Ψ and Φ and corresponding posterior states Ψ°, Φ° after forgetting. Note that at this point $\sqsubseteq$ denotes an appropriate definition of the refinement relation for arbitrary epistemic states with corresponding faithfully assigned total preorders $\preceq_{\Psi}$, $\preceq_{\Phi}$, which could be defined analogously to the refinement relation for OCFs (Def. 2.56). Additionally, it might be interesting to investigate the concept of forgetting conditionals, and if this requires another set of forgetting postulates.

Concerning our research on contractions as a kind of forgetting in Section 4.2, we think that it might be interesting to further investigate the importance of strategic c-contraction in this context. We showed that the assumptions for general c-contractions, which are given by the parameter restrictions defining them (see Def. 3.36), are not sufficient to argue about any of the here presented forgetting postulates **(DFPes-1)**$_{\mathcal{L}}$-**(DFPes-6)**$_{\mathcal{L}}$, except for the success postulate **(DFPes-6)**$_{\mathcal{L}}$. Thus, our further examinations focussed on minimal change c-contractions. However, it might also be appropriate to consider strategic c-contractions that make further assumptions on the parameters without necessarily assuming to correspond to the minimal change paradigm and satisfy the AGM contraction postulates **(AGMes-1)**-**(AGMes-7)**.

More general questions that could be tackled in the future are the influence of forgetting on inductive inference via System Z [Pea90] or c-representations [KI04], the connections between forgetting and irrelevance, and especially which kind of irrelevance we need for forgetting, the concept of remembering, the distributed forgetting in multi agent systems, which corresponds to the idea of collective forgetting [EK19], and forgetting in other concrete epistemic states like probability functions or Markov chains, which would pave the way for transferring the insights from the domain of knowledge representation to the domain of statistical learning.

A Appendix

A.1 Postulates

Let K and K' be belief sets and $\varphi \in \mathcal{L}$ a formula.

(AGM+1) $K + \varphi$ is a belief set

(AGM+2) $\varphi \in K + \varphi$

(AGM+3) $K \subseteq K + \varphi$

(AGM+4) If $\varphi \in K$, then $K + \varphi = K$

(AGM+5) If $K \subseteq K'$, then $K + \varphi \subseteq K' + \varphi$

(AGM+6) $K + A$ is the smallest belief set, such that **(AGM+1)**-**(AGM+5)** hold

Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}$ be formulas and $-$ be a belief change operator.

(AGMes-1) $Bel(\Psi) \models Bel(\Psi - \varphi)$

(AGMes-2) If $Bel(\Psi) \not\models \varphi$, then $Bel(\Psi - \varphi) \models Bel(\Psi)$

(AGMes-3) If $Bel(\Psi - \varphi) \models \varphi$, then $\varphi \equiv \top$

(AGMes-4) $Bel(\Psi - \varphi) \cup \{\varphi\} \models Bel(\Psi)$

(AGMes-5) If $\varphi \equiv \psi$, then $Bel(\Psi - \varphi) \equiv Bel(\Psi - \psi)$

(AGMes-6) $Bel(\Psi - \varphi \wedge \psi) \models Bel(\Psi - \varphi) \vee Bel(\Psi - \psi)$

(AGMes-7) If $Bel(\Psi - \varphi \wedge \psi) \not\models \varphi$, then $Bel(\Psi - \varphi) \models Bel(\Psi - \varphi \wedge \psi)$

Let Ψ and Φ be epistemic states, $\varphi, \psi \in \mathcal{L}$ formulas and $*$ a belief change operator.

(AGMes∗1) $Bel(\Psi * \varphi) \models \varphi$

(AGMes∗2) If $Bel(\Psi) \cup \{\varphi\} \not\equiv \bot$, then $Bel(\Psi * \varphi) \equiv Bel(\Psi) \cup \{\varphi\}$

(AGMes∗3) If $\varphi \not\equiv \bot$, then $Bel(\Psi * \varphi) \not\equiv \bot$

(AGMes∗4) If $\Psi = \Phi$ and $\varphi \equiv \psi$, then $Bel(\Psi * \varphi) \equiv Bel(\Phi * \psi)$

(AGMes∗5) $Bel(\Psi * \varphi) \cup \{\psi\} \models Bel(\Psi * \varphi \wedge \psi)$

(AGMes∗6) If $Bel(\Psi * \varphi) \cup \{\psi\} \not\equiv \bot$, then $Bel(\Psi * \varphi \wedge \psi) \models Bel(\Psi * \varphi) \cup \{\psi\}$

Let Ψ be an epistemic state, $\varphi, \psi \in \mathcal{L}$ formulas and $*$ a belief change operator.

(DP1) If $\varphi \models \psi$, then $Bel((\Psi * \psi) * \varphi) \equiv Bel(\Psi * \varphi)$

(DP2) If $\varphi \models \neg\psi$, then $Bel((\Psi * \psi) * \varphi) \equiv Bel(\Psi * \varphi)$

(DP3) If $Bel(\Psi * \varphi) \models \psi$, then $Bel((\Psi * \psi) * \varphi) \models \psi$

(DP4) If $Bel(\Psi * \varphi) \not\models \neg\psi$, then $Bel((\Psi * \psi) * \varphi) \not\models \neg\psi$

Let $\Gamma, \Gamma' \subseteq \mathcal{L}_\Sigma$ be sets of formulas, P, P' signatures and $\mathcal{F}, \mathcal{F}_O$ as defined in Def. 3.1 and 3.2.

(DFP-1) $\Gamma \models \mathcal{F}(\Gamma, P)$

(DFP-2) If $\Gamma \models \Gamma'$, then $\mathcal{F}(\Gamma, P) \models \mathcal{F}(\Gamma', P)$

(DFP-3) $\mathcal{F}(\Gamma, P) = Cn_{\Sigma \setminus P}(\mathcal{F}(\Gamma, P))$

(DFP-4) If $P' \subseteq P$, then $\mathcal{F}(\Gamma, P) = \mathcal{F}(\mathcal{F}(\Gamma, P'), P)$

(DFP-5) $\mathcal{F}(\Gamma, P \cup P') = \mathcal{F}(\Gamma, P) \cap \mathcal{F}(\Gamma, P')$

(DFP-6) $\mathcal{F}(\Gamma, P \cup P') = \mathcal{F}(\mathcal{F}(\Gamma, P), P')$

(DFP-7) $\mathcal{F}(\Gamma, P) = \mathcal{F}_O(\Gamma, P) \cap \mathcal{L}_{\Sigma \setminus P}$

Let Ψ, Φ be epistemic states over the same signature Σ, $P, P', P_1, P_2 \subseteq \Sigma$ be subsignatures and $\circ_f^\Sigma$ an operator that maps an epistemic state Ψ to another epistemic state Ψ', i.e. $\Psi \circ_f^\Sigma P = \Psi'$.

(DFPes-1)$_\Sigma$ $Bel(\Psi) \models Bel(\Psi \circ_f^\Sigma P)$

(DFPes-2)$_\Sigma$ If $Bel(\Psi) \models Bel(\Phi)$, then $Bel(\Psi \circ_f^\Sigma P) \models Bel(\Phi \circ_f^\Sigma P)$

(DFPes-3)$_\Sigma$ If $P' \subseteq P$, then $Bel((\Psi \circ_f^\Sigma P') \circ_f^\Sigma P) \equiv Bel(\Psi \circ_f^\Sigma P)$

(DFPes-4)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma (P_1 \cup P_2)) \equiv Bel(\Psi \circ_f^\Sigma P_1) \cap Bel(\Psi \circ_f^\Sigma P_2)$

(DFPes-5)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma (P_1 \cup P_2)) \equiv Bel((\Psi \circ_f^\Sigma P_1) \circ_f^\Sigma P_2)$

(DFPes-6)$_\Sigma$ $Bel(\Psi \circ_f^\Sigma P) \equiv Bel((\Psi \circ_f^\Sigma P)_{\uparrow\Sigma}) \cap \mathcal{L}_{\Sigma \setminus P}$

Let Ψ and Φ be epistemic states, $\varphi, \psi \in \mathcal{L}$ be formulas and $\circ_f^\mathcal{L}$ a belief change operator.

(DFPes-1)$_\mathcal{L}$ $Bel(\Psi) \models Bel(\Psi \circ_f^\mathcal{L} \varphi)$

(DFPes-2)$_\mathcal{L}$ If $Bel(\Psi) \models Bel(\Phi)$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \models Bel(\Phi \circ_f^\mathcal{L} \varphi)$

(DFPes-3)$_\mathcal{L}$ If $\varphi \models \psi$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \equiv Bel((\Psi \circ_f^\mathcal{L} \psi) \circ_f^\mathcal{L} \varphi)$

(DFPes-4)$_\mathcal{L}$ $Bel(\Psi \circ_f^\mathcal{L} (\varphi \vee \psi)) \equiv Bel(\Psi \circ_f^\mathcal{L} \varphi) \cap Bel(\Psi \circ_f^\mathcal{L} \psi)$

(DFPes-5)$_\mathcal{L}$ $Bel(\Psi \circ_f^\mathcal{L} (\varphi \vee \psi)) \equiv Bel((\Psi \circ_f^\mathcal{L} \varphi) \circ_f^\mathcal{L} \psi)$

(DFPes-6)$_\mathcal{L}$ If $\varphi \not\equiv \top$, then $Bel(\Psi \circ_f^\mathcal{L} \varphi) \not\models \varphi$

A.2 Proofs

$$\kappa \circ_f^{\Sigma,1} P\ (\omega') = \begin{cases} 0, & \text{if } \kappa_{|\Sigma\setminus P}(\omega') = 0 \\ \max\{\kappa_{|\Sigma\setminus P}\ (\omega) \mid \omega \in \Omega_{\Sigma\setminus P}\} - \kappa_{|\Sigma\setminus P}(\omega') + 1, & \text{otherwise} \end{cases}.$$

Proof. $\circ_f^{\Sigma,1}$ *satisfies* **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**. In order to prove, that $\circ_f^{\Sigma,1}$ satisfies **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**, we first show that $\kappa_{|\Sigma\setminus P}$ and $\kappa \circ_f^{\Sigma,1} P$ result in equivalent beliefs for all $P \subseteq \Sigma$.

$$\begin{aligned} Bel(\kappa \circ_f^{\Sigma,1} P) &\equiv Th(\llbracket \kappa \circ_f^{\Sigma,1} P \rrbracket) && \text{(Lem. 2.39)} \\ &\equiv Th(\{\omega' \in \Omega_{\Sigma\setminus P} \mid \kappa \circ_f^{\Sigma,1} P\ (\omega') = 0\}) && \text{(Def. 2.30)} \\ &\equiv Th(\{\omega' \in \Omega_{\Sigma\setminus P} \mid \kappa_{|\Sigma\setminus P}\ (\omega') = 0\}) \\ &\equiv Th(\llbracket \kappa_{|\Sigma\setminus P} \rrbracket) && \text{(Def. 2.30)} \\ &\equiv Bel(\kappa_{|\Sigma\setminus P}) && \text{(Lem. 2.39)} \end{aligned}$$

In the following, we refer to the above stated equivalence as $(| \equiv \circ_f^{\Sigma,1})$. For all postulates **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**, we that $\circ_f^{\Sigma,1}$ satisfying them can be traced back to the above stated equivalence and the fact that the marginalization satisfies them as well (Th. 4.6).

(DFPes-1)$_\Sigma$:

$$\begin{aligned} & Bel(\kappa) \models Bel(\kappa_{\Sigma\setminus P}) && \text{(Th. 4.6)} \\ \Leftrightarrow\ & Bel(\kappa) \models Bel(\kappa \circ_f^{\Sigma,1} P) && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

(DFPes-2)$_\Sigma$:

$$\begin{aligned} & \text{if } Bel(\kappa) \models Bel(\kappa'),\ \text{then } Bel(\kappa_{\Sigma\setminus P}) \models Bel(\kappa'_{\Sigma\setminus P}) && \text{(Th. 4.6)} \\ \Leftrightarrow\ & \text{if } Bel(\kappa) \models Bel(\kappa'),\ \text{then } Bel(\kappa \circ_f^{\Sigma,1} P) \models Bel(\kappa' \circ_f^{\Sigma,1} P) && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

(DFPes-3)$_\Sigma$:

$$\begin{aligned} & \text{if } P' \subseteq P,\ \text{then } Bel((\kappa_{|\Sigma\setminus P'})_{|(\Sigma\setminus P')\setminus P}) \equiv Bel(\kappa_{|\Sigma\setminus P}) && \text{(Th. 4.6)} \\ \Leftrightarrow\ & \text{if } P' \subseteq P,\ \text{then } Bel((\kappa \circ_f^{\Sigma,1} P') \circ_f^{\Sigma,1} P) \equiv Bel(\kappa \circ_f^{\Sigma,1} P) && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

(DFPes-4)$_\Sigma$:

$$\begin{aligned} & Bel(\kappa_{\Sigma\setminus(P_1\cup P_2)}) \equiv Bel(\kappa_{\Sigma\setminus P_1}) \cap Bel(\kappa_{\Sigma\setminus P_2}) && \text{(Th. 4.6)} \\ \Leftrightarrow\ & Bel(\kappa \circ_f^{\Sigma,1} (P_1 \cup P_2)) \equiv Bel(\kappa \circ_f^{\Sigma,1} P_1) \cap Bel(\kappa \circ_f^{\Sigma,1} P_2) && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

(DFPes-5)$_\Sigma$

$$\begin{aligned} & Bel(\kappa_{\Sigma\setminus(P_1\cup P_2)}) \equiv Bel((\kappa_{\Sigma\setminus P_1})_{(\Sigma\setminus P_1)\setminus P_2}) && \text{(Th. 4.6)} \\ \Leftrightarrow\ & Bel(\kappa \circ_f^{\Sigma,1} (P_1 \cup P_2)) \equiv Bel((\kappa \circ_f^{\Sigma,1} P_1) \circ_f^{\Sigma,1} P_2) && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

(DFPes-6)$_\Sigma$

$$\begin{aligned} & Bel(\kappa_{\Sigma\setminus P}) \equiv Cn_\Sigma(Bel(\kappa_{\Sigma\setminus P})) \cap \mathcal{L}_{\Sigma\setminus P} && \text{(Th. 4.6)} \\ \Leftrightarrow\ & Bel(\kappa \circ_f^{\Sigma,1} P) \equiv Cn_\Sigma(Bel(\kappa \circ_f^{\Sigma,1} P)) \cap \mathcal{L}_{\Sigma\setminus P} && (| \equiv \circ_f^{\Sigma,1}) \end{aligned}$$

□

$$\kappa \circ_f^{\Sigma,2} P\ (\omega') = \begin{cases} 0, & \text{if } \omega' \in \sigma(P)_{|\Sigma\setminus P} \\ \kappa_{|\Sigma\setminus P}(\omega'), & \text{otherwise} \end{cases},\ \sigma(P) = \begin{cases} \bigcup_{\rho\in P} \sigma(\{\rho\}), & \text{if } |P| > 1 \\ \{pbf\}, & \text{if } P = \{p\} \\ \{pb\overline{f}\}, & \text{if } P = \{b\} \\ \{p\overline{b}f\} & \text{if } P = \{f\} \end{cases}$$

Proof. $\circ_f^{\Sigma,2}$ *satisfies* **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**. In order to prove that $\circ_f^{\Sigma,2}$ satisfies **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$**, we first show that the beliefs after marginalizing are equivalent to those after applying $\circ_f^{\Sigma,2}$ to the same OCF κ. We further refer to this equivalence as $(| \models \circ_f^{\Sigma,2})$.

$$\begin{aligned} & Bel(\kappa_{\Sigma\setminus P}) \models Bel(\kappa \circ_f^{\Sigma,2} P) \\ \Leftrightarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \subseteq [\![\kappa \circ_f^{\Sigma,2} P]\!] && \text{(Prop. 2.41)} \\ \Leftrightarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \subseteq [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \end{aligned}$$

Next, we show that $\circ_f^{\Sigma,2}$ satisfies **(DFPes-1)$_\Sigma$-(DFPes-6)$_\Sigma$** due to the above stated relation to the marginalization and the way σ selects the interpretations that are additionally added to the most plausible interpretations.

(DFPes-1)$_\Sigma$:

$$\begin{aligned} & Bel(\kappa) \models Bel(\kappa_{\Sigma\setminus P}) && \text{(Th. 4.6)} \\ \Rightarrow\ & Bel(\kappa) \models Bel(\kappa \circ_f^{\Sigma,2} P) && (| \models \circ_f^{\Sigma,2}) \end{aligned}$$

(DFPes-2)$_\Sigma$: In the following, we assume $Bel(\kappa) \models Bel(\kappa')$.

$$\begin{aligned} & Bel(\kappa \circ_f^{\Sigma,2} P) \models Bel(\kappa' \circ_f^{\Sigma,2} P) \\ \Leftrightarrow\ & [\![\kappa \circ_f^{\Sigma,2} P]\!] \subseteq [\![\kappa' \circ_f^{\Sigma,2} P]\!] && \text{(Prop. 2.41)} \\ \Leftrightarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \subseteq [\![\kappa'_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\ \Leftrightarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \subseteq [\![\kappa'_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\ \Leftarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \subseteq [\![\kappa'_{\Sigma\setminus P}]\!] \\ \Leftrightarrow\ & Bel(\kappa_{\Sigma\setminus P}) \models Bel(\kappa'_{\Sigma\setminus P}) && \text{(Prop. 2.41)} \end{aligned}$$

This holds, since we already know that the marginalization satisfies **(DFPes-2)$_\Sigma$**.

(DFPes-3)$_\Sigma$: In the following, we assume $P' \subseteq P$.

$$\begin{aligned}
& Bel((\kappa \circ_f^{\Sigma,2} P') \circ_f^{\Sigma,2} P) \equiv Bel(\kappa \circ_f^{\Sigma,2} P) \\
\Leftrightarrow\ & [\![(\kappa \circ_f^{\Sigma,2} P') \circ_f^{\Sigma,2} P]\!] = [\![\kappa \circ_f^{\Sigma,2} P]\!] && \text{(Prop. 2.38)} \\
\Leftrightarrow\ & [\![(\kappa \circ_f^{\Sigma,2} P')_{|(\Sigma\setminus P')\setminus P}]\!] \cup \sigma(P)_{|(\Sigma\setminus P')\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\
\Leftrightarrow\ & [\![\kappa \circ_f^{\Sigma,2} P']\!]_{|(\Sigma\setminus P')\setminus P} \cup \sigma(P)_{|(\Sigma\setminus P')\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && \text{(Prop. 3.24)} \\
\Leftrightarrow\ & [\![\kappa \circ_f^{\Sigma,2} P']\!]_{|\Sigma\setminus P} \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && (P' \subseteq P) \\
\Leftrightarrow\ & ([\![\kappa_{\Sigma\setminus P'}]\!] \cup \sigma(P')_{|\Sigma\setminus P'})_{|\Sigma\setminus P} \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\
\Leftrightarrow\ & ([\![\kappa]\!]_{\Sigma\setminus P'} \cup \sigma(P')_{|\Sigma\setminus P'})_{|\Sigma\setminus P} \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && \text{(Prop. 3.24)} \\
\Leftrightarrow\ & [\![\kappa]\!]_{\Sigma\setminus P} \cup \sigma(P')_{|\Sigma\setminus P} \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && (P' \subseteq P) \\
\Leftrightarrow\ & [\![\kappa]\!]_{\Sigma\setminus P} \cup (\bigcup_{\omega\in P'} \sigma(\omega))_{|\Sigma\setminus P} \cup (\bigcup_{\omega\in P} \sigma(\omega))_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\
\Leftrightarrow\ & [\![\kappa]\!]_{\Sigma\setminus P} \cup (\bigcup_{\omega\in P} \sigma(\omega))_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && (P' \subseteq P) \\
\Leftrightarrow\ & [\![\kappa]\!]_{\Sigma\setminus P} \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} \\
\Leftrightarrow\ & [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} = [\![\kappa_{\Sigma\setminus P}]\!] \cup \sigma(P)_{|\Sigma\setminus P} && \text{(Prop. 3.24)}
\end{aligned}$$

(DFPes-4)$_\Sigma$:

$$\begin{aligned}
& Bel(\kappa \circ_f^{\Sigma,2} P_1) \cap Bel(\kappa \circ_f^{\Sigma,2} P_2) \\
\equiv\ & Th([\![\kappa \circ_f^{\Sigma,2} P_1]\!]) \cap Th([\![\kappa \circ_f^{\Sigma,2} P_2]\!]) && \text{(Lem. 2.39)} \\
\equiv\ & Th([\![\kappa_{|\Sigma\setminus P_1}]\!] \cup \sigma(P_1)_{|\Sigma\setminus P_1}) \cap Th([\![\kappa_{|\Sigma\setminus P_2}]\!] \cup \sigma(P_2)_{|\Sigma\setminus P_2}) \\
\equiv\ & Th([\![\kappa_{|\Sigma\setminus P_1}]\!]) \cap Th(\sigma(P_1)_{|\Sigma\setminus P_1}) \cap Th([\![\kappa_{|\Sigma\setminus P_2}]\!]) \cap Th(\sigma(P_2)_{|\Sigma\setminus P_2}) && \text{(Lem. 2.25)} \\
\equiv\ & Bel(\kappa_{|\Sigma\setminus P_1}) \cap Bel(\kappa_{|\Sigma\setminus P_2}) \cap Th(\sigma(P_1)_{|\Sigma\setminus P_1}) \cap Th(\sigma(P_2)_{|\Sigma\setminus P_2}) && \text{(Lem. 2.39)} \\
\equiv\ & Bel(\kappa_{|\Sigma\setminus(P_1\cup P_2)}) \cap Th(\sigma(P_1)_{|\Sigma\setminus P_1}) \cap Th(\sigma(P_2)_{|\Sigma\setminus P_2}) && \text{(Th. 4.6)} \\
\equiv\ & Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)} \cap Th(\sigma(P_1)_{|\Sigma\setminus P_1}) \cap Th(\sigma(P_2)_{|\Sigma\setminus P_2}) && \text{(Prop. 4.5)} \\
\equiv\ & (Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
& \cap (Th(\sigma(P_1)_{|\Sigma\setminus P_1}) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
& \cap (Th(\sigma(P_2)_{|\Sigma\setminus P_2}) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
\equiv\ & (Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
& \cap (\{\varphi \in \mathcal{L}_{\Sigma\setminus P_1} \mid \sigma(P_1)_{|\Sigma\setminus P_1} \models \varphi\} \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) && \text{(Def. 2.22)} \\
& \cap (\{\varphi \in \mathcal{L}_{\Sigma\setminus P_2} \mid \sigma(P_2)_{|\Sigma\setminus P_2} \models \varphi\} \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
\equiv\ & (Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
& \cap \{\varphi \in \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} \mid \sigma(P_1)_{|\Sigma\setminus P_1} \models \varphi\} \\
& \cap \{\varphi \in \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} \mid \sigma(P_2)_{|\Sigma\setminus P_2} \models \varphi\} \\
\equiv\ & (Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
& \cap \{\varphi \in \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} \mid \sigma(P_1)_{\Sigma\setminus(P_1\cup P_2)} \models \varphi\} && \text{(Lem. 3.27)} \\
& \cap \{\varphi \in \mathcal{L}_{\Sigma\setminus(P_1\cup P_2)} \mid \sigma(P_2)_{\Sigma\setminus(P_1\cup P_2)} \models \varphi\}
\end{aligned}$$

$$
\begin{aligned}
&\equiv (Bel(\kappa) \cap \mathcal{L}_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\quad \cap Th(\sigma(P_1)_{\Sigma\setminus(P_1\cup P_2)}) && \text{(Def. 2.22)} \\
&\quad \cap Th(\sigma(P_2)_{\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Bel(\kappa_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\quad \cap Th(\sigma(P_1)_{\Sigma\setminus(P_1\cup P_2)}) && \text{(Prop. 4.5)} \\
&\quad \cap Th(\sigma(P_2)_{\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket) \\
&\quad \cap Th(\sigma(P_1)_{\Sigma\setminus(P_1\cup P_2)}) && \text{(Lem. 2.39)} \\
&\quad \cap Th(\sigma(P_2)_{\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup \sigma(P_1)_{\Sigma\setminus(P_1\cup P_2)} \cup \sigma(P_2)_{\Sigma\setminus(P_1\cup P_2)}) && \text{(Lem. 2.25)} \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup (\sigma(P_1) \cup \sigma(P_2))_{\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup ((\bigcup_{\rho\in P_1} \sigma(\rho)) \cup (\bigcup_{\rho\in P_2} \sigma(\rho)))_{\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup (\bigcup_{\rho\in P_1\cup P_2} \sigma(\rho))_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup \sigma(P_1 \cup P_2)_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa \circ_f^{\Sigma,2} (P_1 \cup P_2) \rrbracket) \\
&\equiv Bel(\kappa \circ_f^{\Sigma,2} (P_1 \cup P_2)) && \text{(Lem. 2.39)}
\end{aligned}
$$

(DFPes-5)$_\Sigma$:

$$
\begin{aligned}
&Bel((\kappa \circ_f^{\Sigma,2} P_1) \circ_f^{\Sigma,2} P_2) \\
&\equiv Th(\llbracket (\kappa \circ_f^{\Sigma,2} P_1) \circ_f^{\Sigma,2} P_2 \rrbracket) && \text{(Lem. 2.39)} \\
&\equiv Th(\llbracket (\kappa \circ_f^{\Sigma,2} P_1)_{|(\Sigma\setminus P_1)\setminus P_2} \rrbracket \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) \\
&\equiv Th(\llbracket \kappa \circ_f^{\Sigma,2} P_1 \rrbracket_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) && \text{(Prop. 3.24)} \\
&\equiv Th((\llbracket \kappa_{|\Sigma\setminus P_1} \rrbracket \cup \sigma(P_1)_{|\Sigma\setminus P_1})_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) \\
&\equiv Th((\llbracket \kappa \rrbracket_{|\Sigma\setminus P_1} \cup \sigma(P_1)_{|\Sigma\setminus P_1})_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) && \text{(Prop. 3.24)} \\
&\equiv Th(\llbracket \kappa \rrbracket_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_1)_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_1)_{|(\Sigma\setminus P_1)\setminus P_2} \cup \sigma(P_2)_{|(\Sigma\setminus P_1)\setminus P_2}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|\Sigma\setminus(P_1\cup P_2)} \cup \sigma(P_1)_{|\Sigma\setminus(P_1\cup P_2)} \cup \sigma(P_2)_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|\Sigma\setminus(P_1\cup P_2)} \cup (\sigma(P_1) \cup \sigma(P_2))_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|\Sigma\setminus(P_1\cup P_2)} \cup ((\bigcup_{\rho\in P_1} \sigma(\rho)) \cup (\bigcup_{\rho\in P_2} \sigma(\rho)))_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|\Sigma\setminus(P_1\cup P_2)} \cup (\bigcup_{\rho\in P_1\cup P_2} \sigma(\rho))_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa \rrbracket_{|\Sigma\setminus(P_1\cup P_2)} \cup \sigma(P_1 \cup P_2)_{|\Sigma\setminus(P_1\cup P_2)}) \\
&\equiv Th(\llbracket \kappa_{|\Sigma\setminus(P_1\cup P_2)} \rrbracket \cup \sigma(P_1 \cup P_2)_{|\Sigma\setminus(P_1\cup P_2)}) && \text{(Prop. 3.24)}
\end{aligned}
$$

$$\equiv Th([\![\kappa \circ_f^{\Sigma,2} (P_1 \cup P_2)]\!])$$
$$\equiv Bel(\kappa \circ_f^{\Sigma,2} (P_1 \cup P_2)) \qquad \text{(Lem. 2.39)}$$

(DFPes-6)$_\Sigma$:

$$Cn_\Sigma(Bel(\kappa \circ_f^{\Sigma,2} P)) \cap \mathcal{L}_{\Sigma\setminus P}$$
$$\equiv \{\varphi \in \mathcal{L}_\Sigma \mid Bel(\kappa \circ_f^{\Sigma,2} P) \models \varphi\} \cap \mathcal{L}_{\Sigma\setminus P} \qquad \text{(Def. 2.16)}$$
$$\equiv \{\varphi \in \mathcal{L}_{\Sigma\setminus P} \mid Bel(\kappa \circ_f^{\Sigma,2} P) \models \varphi\}$$
$$\equiv Cn_{\Sigma\setminus P}(Bel(\kappa \circ_f^{\Sigma,2} P))$$
$$\equiv Bel(\kappa \circ_f^{\Sigma,2} P) \qquad (Bel \text{ is deductively closed})$$

□

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